COMPARISON OF THE RIEMANN AND LEBESGUE INTEGRAL

CHOW LEE KUM

FAKULTI SAINS DAN TEKNOLOGI
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COMPARISON OF THE RIEMANN AND LEBESGUE INTEGRAL

by
CHOW LEE KUM

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Adalah ini diakui dan disahkan bahawa laporan penyelidikan bertajuk "Comparison of the Riemann Integral and Lebesgue Integral" by Chow Lee Kum, No. Matriks: UK 14462 telah diperiksa dan semua pembetulan yang disarankan telah dilakukan. Laporan ini dikemukakan kepada Jabatan Matematik sebagai memenuhi sebahagian daripada keperluan memperolehi Ijazah Sarjana Muda Sains Matematik Komputasi, Fakulti Sains dan Teknologi, UMT.

Disahkan oleh:

....

Penyelia Utama

Nama: Dr. Zabidin bin Salleh

Cop Rasmi:

DR. ZABIDIN BIN SALLEH
Pensyarah
Jabatan Matematik
Fakulti Sains dan Teknologi
Universiti Malaysia Terengganu
21030 Kuala Terengganu

Tarikh: 6/5/2009

....

Ketua Jabatan Matematik

Nama: Dr. Haji Mustafa Bin Mmamat

Cop Rasmi:

DR. HJ. MUSTAFA BIN MAMAT
Ketua
Jabatan Matematik
Fakulti Sains dan Teknologi
Universiti Malaysia Terengganu
21030 Kuala Terengganu

Tarikh: 6/5/09

DECLARATION

I hereby declare that this final year project entitled "Comparison of the Riemann Integral and Lebesgue Integral" is the result of my own research except as cited in the references.

Signature :
Name : Chow Lee Kum
Matrix No : UK14462
Date : 6th May 2009

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COMPARISON OF THE RIEMANN INTEGRAL AND LEBESGUE INTEGRAL

ABSTRACT

The development of the integral is most introductory analysis course is centered almost exclusively on the Riemann integral. In this historical development the integration is simply introduced as finding the area under a curve. The Riemann integration is a basic concept in mathematical analysis, since it related to boundedness, continuity and differentiability. We also consider some integrals of Stieltjes types which are considered as generalization of the Riemann Integrals which involves two bounded functions. The Stiltjes integral has very useful applications in probability theory, mechanics as well as theoretical physics. Another theory of integration more general than the Riemann theory was called Lebesgue integral, it consider the concept of measure of a set, starting with simple function and ending with measurable function, this approach leads to greater generality in the types of function that can integrated. We will compare both of this integration by using their theorem.

PERBANDINGAN ANTARA PENGAMIRAN RIEMANN DAN PENGAMIRAN LEBESGUE

ABSTRAK

Dalam bidang analisis, kamiran Riemann adalah pendahuluan yang paling istimewa dalam perkembangan pengamiran. Dalam perkembangan lepas, pengamiran hanya semata-matanya untul memperkenalkan mencari luas dibawah satu lengkumg. Kamiran Riemann adalah konsep asas dalam analisis matematik, ia dikaitkan dengan keterbatasan, keselanjaran, dan kebolehbezaan. Kita juga menimbangkan sesetengah kamiran iaitu model Stieltjes di mana mengangap sebagai generalisasi daripada kamiran Rieamann dimana ia melibatkan dua fungsi batas. Stieltjes sangat berguna dalam aplikasi dalam teori kebarangkalian, mekanik seperti berdasarkan teori fizik. Theori pengamiran yang seturusnya adalah lebih umum daripada kamiran Riemann adalah dinamakan sebagai kamiran Lebesgue, ia dipertimbangkan sebagai konsep pengukuran suatu set, dimulakan dengan fungsi mudah dan measurable fungsi sebagai pengakhiran, pencapaian ini memimpin kamiran yang lebih baik untuk pelbagai fungsi. Kita akan membandingkan kedua-dua pengamiran tersebut dengan mengguankan prinsip yang telah dibuktikan secara logik.

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NOTATION

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Symbol	*Meaning for*
$\emptyset$	empty set
$x \in S$	x is an element of S
$x \notin S$	x is not an element of S
$\{x\}$	set having x as unique element
$\{x : ...\}$	set of all element with the property
$X \bigcup Y$	union of the set X and Y
$X \bigcap Y$	intersection of the set X and Y
$X \subseteq Y$	set x is contained in the set Y; set x is subset of Y.
$X \subset Y$	set x is proper subset of Y.
$X \setminus Y$	set of all elements that live in X but not in Y.
$\Rightarrow$	implies (gives)
$\Leftrightarrow$	if and only iff; or "iff"
$\rightarrow$	converges to; into
N	set of natural number, $\{1, 2, ...\}$
Z	set of all integer, $\{...,-2,-1,0,1,2,...\}$
Q	set of all rational number $\left\{\frac{p}{q} : p.q \in z; q \neq 0\right\}$
R	set of all real number
C	set of all complex number
$\limsup \mid z_n \mid$	upper limit of the real sequence., $\{ \mid z_n \mid \}$
$\liminf \mid z_n \mid$	lower limit of the real sequence., $\{ \mid z_n \mid \}$
$\lim \mid z_n \mid$	limit of the real sequence., $\{ \mid z_n \mid \}$
Sup S	least upper bound, or the suoremum, of the set $S \subset R$
Inf S	greatest lower bound, or the suoremum, of the set $S \subset R$
$f : D \rightarrow D_1$	f is the function from D to D_1
f(x)	the value of the function at x.
$f \circ g$	composition mapping of f and g
$\sup_{x \in D} f(x)$	supremum of f in D
$\inf_{x \in D} f(x)$	infimum of f in D
$[x_1, x_2]$	$\{x \in R : x_1 \leq x \leq x_2\}$
(x_1, x_2)	$\{x \in R : x_1 < x < x_2\}$

Symbol	*Meaning for*
Δ	$\Delta(0;1)$, unit disc $\{z \in C : \lvert z \rvert < 1\}$
$f^{(n)}(x)$	*n-th* derivative of *f* at *x*.
$\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$	sequence $\{x_n\}$ converges to *x* with a matric *d*.

CHAPTER 1

INTRODUCTION

The term of the integration will not meaning finding whose derivatives is known. This process will be referred to as "antidifferentiation". Thus a function f is an antiderivatives of f', and when the domain of f is an interval, any other antiderivatives of f' must be of the form $f+C$, where C is a constant function. Under certain broad conditions, the integral of a function can be evaluated via its antiderivatives.

1.1 Introduction of the Riemann Integral

The Riemann integration is a basic concept in mathematical analysis, since it related to boundedness, continuity and differentiability. The Riemann integral was started from the following problem.

The problem of finding the area of a plane region bounded by vertical lines $x=a$ and $x=b$, the horizontal line $y=0$, and the graph of the non-negative function $y=f(x)$, is a very old one. The Greeks has a method which they applied successfully to simple cases such as $y=x^2$. This "method of exhaustion" consisted essentially in approximating the area by figures whose areas were known as rectangles and triangles. Then an appropriated limit was taken to obtain the result.

In the 17th century, Newton and Leibnitz independently found an easy method for solving the problem, both consider integration as the inverse operation of differentiation. For example, in the De analysis, Newton proved that the area under the curve $y = ax^{m/n}$ $(m/n \neq -1)$ is given by $\frac{an\,x^{m/n+1}}{m+n}$ by using his differential calculus to prove that if $A(x)$ represents the area from 0 to x then $A'(x) = ax^{m/n}$. Even though Leibniz arrived at the concept of the integral by using sums to compute the area, integration itself was always the inverse operation of differentiation. The definite integral of a function $f(x)$ on $[a, b]$, denoted by $\int_a^b f(x)dx$ and the area is given by $F(b)$-$F(a)$, where F is an antiderivative of f. This is the familiar Fundamental Theorem of Calculas; it reduced the problem of finding areas to that of finding antidrivatives. Its attention was focused on the inverse character of differentiation and techniques of evaluating both definite and indefinite integrals. This remained the definition of the definite integral until the 1820s.

Eventually mathematicians began to worry about the function not having antiderivatives. When that happened, they were forced to return again to the basic problem of area. At the same time, it becomes clear that a more precise formulation of the problem is necessary. Exactly what is area, anyway? Or more generally, how can $\int_a^b f(x)dx$ be defined rigorously for all continuous functions? This was approach to integration is due to Cauchy, who was the first mathematicians to construct a theory of integration based on approximating the area under the curve. By Cauchy's first method, and then for functions that may have a finite number of discontinuities at which the function is unbounded, by Cauchy's second method. Cauchy's first method can be applied to all continuous functions, all bounded functions with finitely many discontinuities, and it can be applied to some bounded functions with infinitely many discontinuities.

In the middle of the 19^{th} century, Cauchy and Riemann put the theory of integration on a firm footing. They described that at least theoretically on how to carry out the programs of the Greeks for any function f. The result was the definition of what is now called the Riemann integral of f. The Riemann integral, proposed by Georg Bernhard Riemann (1862-1866), is a broadly successful attempt to provide such as foundation for the integer. Riemann was led to the development of the integral by trying to characterize which functions were integrable according to Cauchy's definition. In the process, he modified Cauchy's definition and developed the theory of integration that bears his name. One of his achievements was providing necessary and sufficient conditions for a real-value bounded function to be integrable. Riemann's definition starts with the construction of a sequence problem, and gives useful results for many other problems. The unbounded functions that are extended by Cauchy, the concept were resulted in a complete and formal expression as a limit of the certain sum. He concluded that function which is not covered by Dirichlet does not exist in nature. But there were new applications of trigonometric series to number theory and other places in pure mathematics. This provided impetus to pursue these foundational questions. Bernhard Riemann assumed that

$$\lim_{\|P\|\to 0}(D_1\delta_1 + D_2\delta_2 + \ldots + D_n\delta_n) = 0 .$$

Where P is a partition of $[a, b]$ with δ_i the length of the subintervals and the D_i are corresponding oscillates of $f(x)$:

$$D_i = |\sup_{x\in I} f(x) - \inf_{x\in I} f(x)|$$

For a given P and $\delta > 0$, define

$$S = s(P,\delta) = \sum_{D_i > \delta} \delta_i .$$

The Riemann-Stieltjes integral, proposed were developed by Thomas-Jean Stieltjes (1856-1894) and arise in many applications in both mathematics and physics. The general Riemann-Stieltjes integral that will give meaning of the following types of integrals:

$$\int_0^1 f(x)dx^2, \qquad \int_a^b f(x)d[x] \quad \text{or} \quad \int_a^b f(x)d\alpha(x)$$

where α is a monotone increasing function on $[a, b]$. The Riemann-Stieltjes integral permits the expression of many seemingly diverse results as a single formula.

The beginning of this century saw the development of the notion of the measure of a set of real numbers that paved the way to the foundations of the modern theory of the Lebesgue integral. Now was the evitable generalization of the Riemann integral.

1.2 Introduction of the Lebesgue Integral

The Lebesgue integral using the concept of measure plays an important role in the branch of mathematics called real analysis and in many other fields in the mathematical sciences. On the real line, the idea of measure generalizes the length of an interval, in the plane, the area of a rectangle, and forth, this links us to measure of a set. It allows us to talk about the measure of a set in the same way that we talk about the length of an interval. The development of the Riemann integral of a bounded function on a closed and bounded interval depend on the partitioned $[a, b]$ into interval. But the time of Riemann there was only an imperfect understanding of sets of real numbers and so it did not occur to Riemann that the property of Riemann integrability for a bounded function f depended exclusively on the nature of the set of points of discontinuity of f. The notion of measure and measurable set play a prominent role in the development in the Lebesgue integral in that we partitioned $[a, b]$ not into interval, instead into pairwise disjoint measurable sets.

The Lebesgue integral is named from Henri Lebesgue (1875-1941), a French mathematicians, who introduced and complete the theorem by using the notion of a set of measure zero which is the integral in (Lebesgue 1904) and defined measure of subsets of the line and the plane, as well as the Lebesgue integral of a nonnegative function. The term "Lebesgue integration" may refer either to the general theory of integration of a function with respect to a general measure, as introduced by Lebesgue, or to the specific case of integration of a function defined on a sub-domain of the real line with respect to Lebesgue measure. Like Riemann, Lebesgue was led to the development of his theory of integration while searching for sufficient condition on a function f for which the integrals defining the Fourier coefficients of f exists.

In mathematics, the integral of a non-negative function can be regarded in the simplest cases as the area between the graph of that function and the x-axis. Lebesgue integration is a mathematical construction that extends the integral to a larger class of function. It also extends the domains on which these functions can be defined. It has long been understood that for non-negative function with continuous functions on closed bounded interval graph, the area under the curve could be defined as the integral. However, as the need to consider more irregular functions arose it became clear that more careful approximation techniques would be needed in order to define a suitable integral.

Lebesgue was exhibited a trigonometric series that converges everywhere to a nonnegative function f that was not Riemann integrable. The function f, however, is integrable according to Lebesgue's definition and the trigonometric series is the Fourier series of f. Lebesgue's theory of integration allows us to prove interchange of limit and integrations theorems without requiring uniform convergence of the sequence of functions.

1.3 Problem Declaration

Although the Riemann integral suffices in most daily situation, it fails to meet our needs in several important ways. First, the class of Riemann integrable functions is relatively small. Second and related to the first, the Riemann does not have satisfactory limits properties. That is, given a sequence of Riemann integrable function $\{f_n\}$ with a limit function $f = \lim_{n\to\infty} f_n$, it does not necessarily follow that the limit function f is Riemann integrable. Lastly, partitioning the range of a function and counting the resultant rectangles becomes tricky since we must employ some way of determining how much of the domain is sent to a particular portion of a partition of the range.

1.4 Research Objective

The objective of this final year project is to:

i. Compare the difference between Riemann integral and Lebesgue integral.
ii. A direct theory of the Riemann and Lebesgue integral also need to develope.
iii. By compare both of the integral, we must consider the interval and the convergence of the function.
iv. Aside from examining the convergence properties of the Lebesgue integral, we are also interest in how it behaves relative to the Riemann integral.

1.5 Scope Research

The Riemann integral simply introduced as finding the area under a curve it related to boundedness, continuity and differentiability. We also consider the Riemann-Stieltjes which involves two bounded functions. Rather than partitioning the domain of the function, as in the Riemann integral, we use the Lebesgue to partition the range. Thus, for each interval in the partition, rather than asking for the value of the function between the end points of the interval in the domain, how much of the domain is map by the function to some value between two ends point in the range is consider. Aside from examining the convergence properties of the Lebesgue Integral, we are also interested in how it behaves relative to the Riemann Integral.

CHAPTER 2

LITERATURE REVIEW

According to the Cerone, P and Dragomir, S (2008) had already proved that the approximating of the Riemann-Stieltjes integral via some moments of the integrand. It started from the simple expression of the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ was that $\frac{u(b)-u(a)}{b-a}\int_a^b f(t)d(t)$, when it existed then $|D(f,u;a,b)| \leq \frac{1}{2}L(M-m)(b-a)$ held. L-Lipschitzian of the integrand f of the Riemann integral was $|u(t)-u(s)| \leq L|t-s|$ for each $t,s \in [a,b]$. The expression that contained the function *f*, $\frac{p}{(b-a)^p}\left[u(b).\int_a^b (t-a)^{p-1} f(t)dt - u(A).\int_a^b (b-t)^{p-1} f(t)dt\right]$ of the integral exists where *p*>o and this general result could be proved by the integration by part. For the further bounds for monotonic integrand, the integrator f was monotonic non-decreasing for the error functional *F*(*f*,*u*,*p*;*a*,*b*). At last approximating the finite Fourier transform of $f:[a,b] \to R$ on the finite interval [*a*, *b*] and *f*(*g*) was $F(t) := \int_a^b f(s)e^{-2\pi its}$ to provide a composite rule in approximating the Finite Fourier Transform in terms of moments for the function f and the quadrature rule.

Jack, G (2007) had introduced Random Riemann Integral. It could be done by using Riemann sums which is random variables. However, the idea of the random Riemann integral came from the first return integral. It was a sequence of real numbers that dense in the unit interval and belonged to the first return points of the interval. This integral was further considered as random variables, it followed the Lebesgue measurable function, f from the unit interval $I := [0,1]$ into R. For the Random Riemann sums, it could be defined the Random Riemann sums of f on P if the integral exists. And it was used that $E\sum | x_k |^p = \sum | I_k |^{p-1} \cdot \int_{I_k} | f |^p \leq | p |^{p-1} \int | f |^p$.

The Random Riemann sums of a function converged in probability to its Lebesgue integral would give a sequence of partitions whose size tended to zero. The convergence of the function depended on the size of the partition in the sequences but it was used a different construction to choose the random points. This almost proved the almost sure convergence of random Riemann sums to the Lebedgue integral.

Enrigue, A (1986) developed a very direct theory of the Lebesgue Integral with the title of The Lebesgue Integral as a Riemann Integral. A general definition of the Lebesgue Integral is preceded by definitions for simple functions, then for bounded functions over a set of finite measure and then for nonnegative functions, a step function and then for upper functions, and again a certain amount of theory is developed in each particular case. The Lebesgue integral as a Riemann integrals is defined to allow for complete generality. A theory for functions defines on a set of finite measure is makes uses to Riemann-Stieltjes Integral. To define the theorem of the Lebesgue measure and measurable functions, a rectangle in R^n with the product space n are bounded intervals, open, closed, or neither. The volume of such a rectangle R, denoted by $m(R)$, is the products of the lengths of its component intervals. The Lebesgue approach to define the integral of a function $f : A \subset R^n \to R$ is to partition its range, not its domain as in the Riemann theory. The convergence theorem of the Lebesgue Integral used the validity of the limit $\int_A f_N \to \int_A f$ when $f_N \to f$. In the case of Riemann integration the uniform convergence of this sequence is a sufficient condition. And this allows used to prove the theorem of the measurable sets and functions.

According to Robert, G (1966), he stated that the characteristic of a given set A is a very simple function and contains only two elements, 0 and 1. The Riemann integral is undefined for the function. The characteristic function is only one function in a class of functions that are not Riemann integrable. In the beginning of the 20th century, Henri Lebesgue, developed the theory of Lebesgue integration and measure to opened up a larger class of functions over which an integral can be defined and calculated – including the very basic characteristic function. The definition of the Lebesgue integral mirrors that of the Riemann integral where one takes the infimum and supremum over all approximating step functions. The fundamental difference between the Riemann integral and the Lebesgue integral is that any function that is Lebesgue measurable is Lebesgue integrable; whereas measurability is not a sufficient condition for Riemann integrability. Lebesgue integrable is larger than the class of functions which are Riemann integrable. In the theory of Lebesgue integral, measurability replaces the need for complete continuity because many sets and functions are measurable. Lebesgue is able to build a theory of integration and its properties over a very large class of functions with the notion of almost everywhere. Lebesgue Dominated Convergence Theorem states that if a sequence of integrable functions f_n converge almost everywhere to a real-valued measurable function f where $|f(x)| \leq g(x)$, another real-valued integrable function, then f is integrable and

$$\int f\, d\mu = \lim \int f_n\, d\mu$$

where μ denotes the Lebesgue measure function. Convergence of measurable f_n to measurable f in the Dominated Convergence Theorem can be shown that if a sequence of measurable f_n in L_p converges almost everywhere to a measurable f and $|f_n(x)| \leq g(x)$, a measurable function, for all N and x, then f belongs to L_p and f_n converges to f in L_p. A sequence of measurable real-valued f_n is said to converge in measure to a measurable real-valued f if

$$\lim_{n\to\infty} \mu(\{x \varepsilon X : |f_n(x) - f(x)| \geq a\}) = 0\,.$$

A sequence can be defined to be Cauchy in measure in a similar way. Also f_n uniformly converges to f if the set

$$\{x \varepsilon X : | f_n(x) - f(X) | \geq a\}$$

is the empty set. If f_n converges uniformly, then f_n converges in measure. Another theorem stemming from the notion of convergence in measure is the convergence of subsequences of f_n. The theorem states that if there is sequence of measurable real-valued f_n which is Cauchy in measure, then there is a subsequence of measurable real-valued functions that converge almost everywhere and in measure to a measurable real-valued f. Another corollary states that if a sequence of measurable real-valued f_n is Cauchy in measure, then f_n converges in measure to a measurable real-valued function f for which its limit function is uniquely determined almost everywhere. Finally the Lebesgue Dominated Theorem shows that if f_n is a sequences of measurable functions in L_p which converge in measure to a measurable f and if $| f_n(x) | \leq g(x)$ (a measurable function in L_p) almost everywhere, then f is in L_p and f_n converges in L_p to f. An important aspect of the Lebesgue theory is the goal of defining a more universal notion of length, (measure) which is to increase the class of Riemann integrable functions to what has been defined as the class of Lebesgue integrable functions. For more abstract domains, definition of the outer measure μ^* is define as $\mu^*(B) = \inf \sum_{j=1} \mu(E_j)$. With the notion of outer measure, on any subset of R_n, one can restrict the Lebesgue measure μ to μ*, the outer measure. Hence μ(E)= μ*(E). As a conclusion, there are sets and functions that are not Lebesgue measurable or integrable. Nevertheless, the study of the Lebesgue theory is very useful. The number of functions which one is able to integrate with Lebesgue is quite large and contains a great deal of functions which the Riemann integral has no way of exploring. Hence the Lebesgue theory of integration and measurability is a very powerful result in analysis upon which mathematicians can built upon.

There are the relations between The Fundamental Theorem of Calculus for Lebesgue Integral. There are following theorem are stated about the topic.

Theorem 2.5: A function $f:[a,b] \to R$ is absolutelt continuous if and only if it is differentiable almost everywhere, its dirivatives $f' \in L^1[a,b]$ and, for each $t \in [a,b]$,

$$f(t) = f(a) + \int_a^t f'(s)ds\,.$$

Theorem 2.6 (Lebesgue Differentiation Theorem)**:** Every bounded variation function $f:[a,b] \to R$ is differentiable almost everywhere with derivatives belonging to $L^1[a,b]$. If the function f is non-decreasing. Then

$$\int_a^t f'(s)ds \le f(b) - f(a)$$

Theorem 2.7: If $f:[a,b] \to R$ is absolutely continuous with $f' = 0$ almost everywhere then f is constant.

Theorem 2.8: Every bounded variation function $f:[a,b] \to R$ determines a unique Lebesgue-Stieltjes measure η. The function f is absolutely continuous if and only if its corresponding Lebesgue-Stieltjes measure η is absolutely continuous with respect to Lebesgue measure.

Theorem 2.9: If $f:[a,b] \to R$ is a bounded variation function with associated Lebesgue-Stieltjes measure η, then the following statements are equivalent:

a) f is differentiable at x and $f'(x) = A$.

b) For each $\varepsilon > 0$ there is $\delta > 0$ such that $\left|\frac{\eta(I)}{m(I)} - A\right| < \varepsilon$, wherever I is an open interval with Lebesgue measure $m(I) < \delta$ and $x \in I$

CHAPTER 3

THE RIEMANN INTEGRAL

In this chapter, we will define the Riemann integral and give a detailed and rigorous account of Riemann integration, proving the basic property of integration as anti-derivative which comes out as the fundamental theorem of calculus.

3.1 The Riemann Integral

The "inverse" operation of differentiation is integration. We use the integral of a function to get the area under the curve:

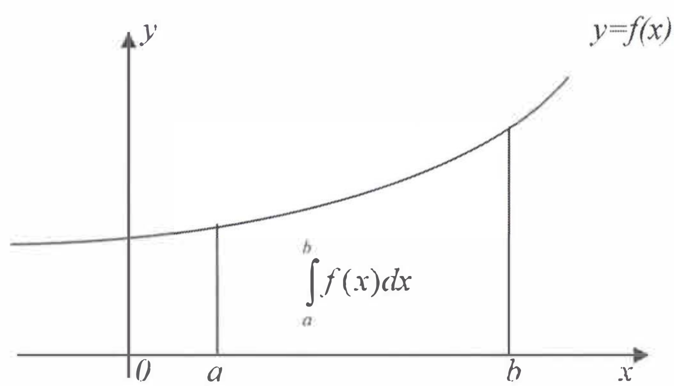


Figure1: The definite integral of *f(x)* between "*a*" and "*b*"

The function of $\int_a^b f(x)dx$ is being integrated. The number of "*a*" and "*b*" are the lower and upper limits of the integral to define the area under the curve starts and ends. The "*dx*" was the integrating with respect to *x*. Figure 1 is a curve and so changes as *x* changes. Up to this point, the only methods for calculating area are we know are for simple geometric shapes, particularly rectangles:

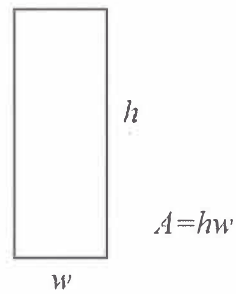


Figure 2: Area of a rectangle= heights x width

We will use rectangles to figure our area, and we also add more rectangles to make the area more accurate. The approximation is very important because it can be refined. We will make better refinements to the area by adding more rectangles until we have something we can use as a limit. We have two choices for the height of the rectangle, the minimum value and the maximum value of *f(x)* for x in [a, b]:

a) If the minimum value for *f(x)* was choose, that is *f(a)*:

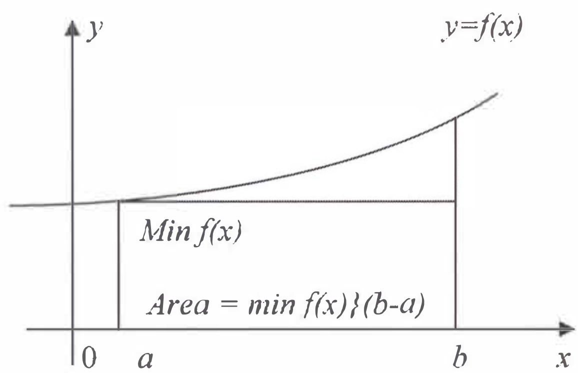


Figure 3: First approximation of integral using minimum value of *f*(*x*)

The width of the rectangle is the difference between the endpoints, $w = b - a$, and the area approximation is

$$L(f, P) = \min_{x\in[a,b]} f(x)(b-a) = f(a)(b-a).$$

L was represents the area approximation which are approximate from below and its depends on the function *f(x)* and the partition P. The approximation is less than the real area. This is the first approximation and so P is just the interval and the value of this approximation, $f(a)(b-a)$ are well-defined.

b) If instead the maximum value of $f(x)$ over the interval as the height of the rectangle, that is $f(b)$,

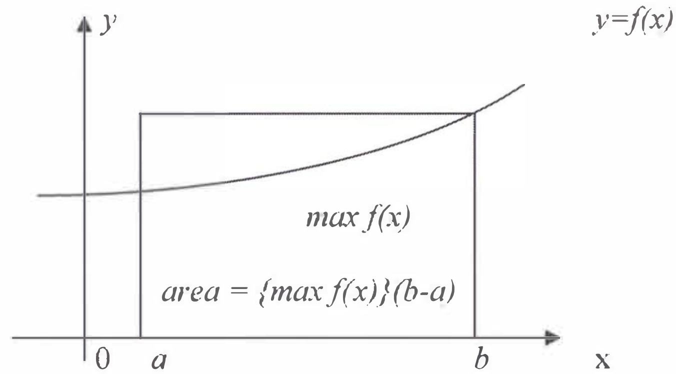


Figure 4: First approximation of integral using maximum value of $f(x)$

the approximation will becomes:

$$U(f,P) = \max_{x\in[a,b]} f(x)(b-a) = f(b)(b-a).$$

U was represents the area approximation and the maximum of $f(x)$ and the width of the interval was over lined to emphasize that this estimate overshoots the real area. It is obvious that

$$L(f,P) \le \int_a^b f(x)dx \le U(f,P).$$

Definition 3.1.1: Suppose f is a bounded real-valued function, given R is a closed and bounded interval, $[a, b]$, $a < b$. a finite set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_n = b$ is the partition P of $[a, b]$. To improve the estimate, we cut this interval into two equal pieces:

$$P = \{P_1, P_2\} = \left\{\left[a, a+\frac{b-a}{2}\right], \left[a+\frac{b-a}{2}, b\right]\right\} = \{[x_0, x_1], [x_1, x_2]\}.$$

The basic idea of a partition is to divide the interval [a ,b] into a finite collection of subintervals. Specifically, we have $n+1$ points of division, with the first point being $x_0 = a$ and the last point being $x_n = b$. There is $\Delta x_i = x_i - x_{i-1}$. $i = 1, 2, \ldots, n$ which is equal to the length of the interval $[x_{i-1}, x_i]$, this call i-th subintervals and clearly non-overlapping.

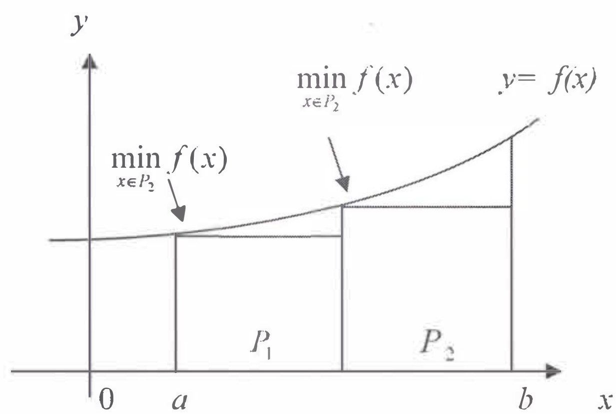


Figure 5: Second approximation using minimum value of $f(x)$

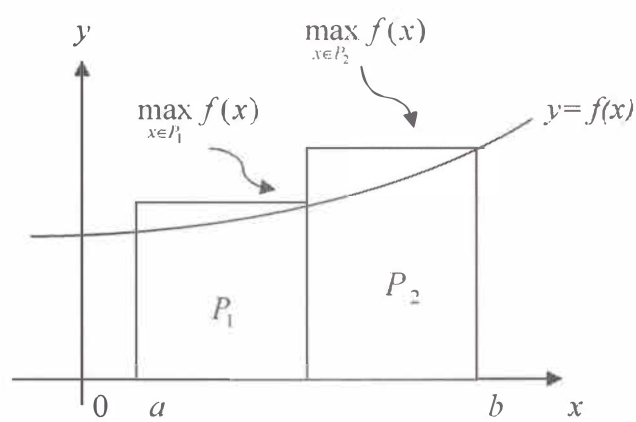


Figure 6: Second approximation using maximum value of $f(x)$

Now, we have

$$L(f,P) = \min_{x \in [x_0, x_1]} f(x)(x_1 - x_0) + \min_{x \in [x_1, x_2]} f(x)(x_2 - x_1) = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1)$$

and

$$U(f,P) = \max_{x \in [x_0, x_1]} f(x)(x_1 - x_0) + \max_{x \in [x_1, x_2]} f(x)(x_2 - x_1) = f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1).$$

The real areas are still between in these two estimates. We will refined our estimates by chopping up the interval [a, b] into ever-smaller pieces and calculate the upper and lower estimated each time. Suppose that we done in n times, each piece of the interval has the same width.

$$\Delta x = x_{i+1} - x_i = \frac{b-a}{n}, \qquad k=0, 1, \ldots, n.$$

The functions $f(x)$ we have shown in the graph is increasing, so the minimum for each piece is at the far left end of the piece while the maximum is at the far right end. Let f be a real-valued function defined on $[a, b]$ and bounded. We will let

$$m = \inf\{f(x) : x \in [x_{i-1}, x_i]\},$$

$$M = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Sum of the integral is defining using supremum principle which is realized as the limit of a set of suitable sums. And the *inf* is defined as the infimum.

Definition 3.1.2: The upper sum $U(P, f)$ for the partition P and function f is defined by

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

and the lower sum

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i ,$$

since $m_i \le M_i$, for all $i = 1, \ldots, n$, then $L(P, f) \le U(P, f)$ for any partition P of $[a, b]$.

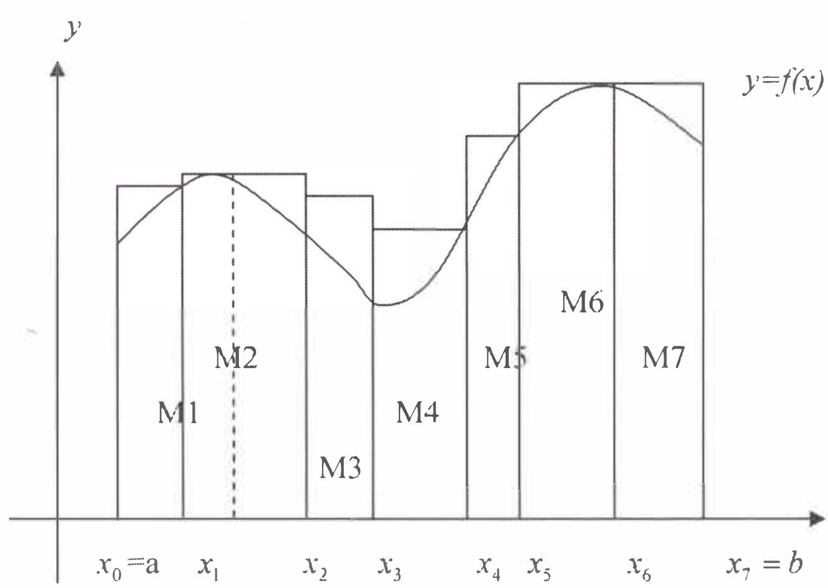


Figure 7: $U(P, f)$

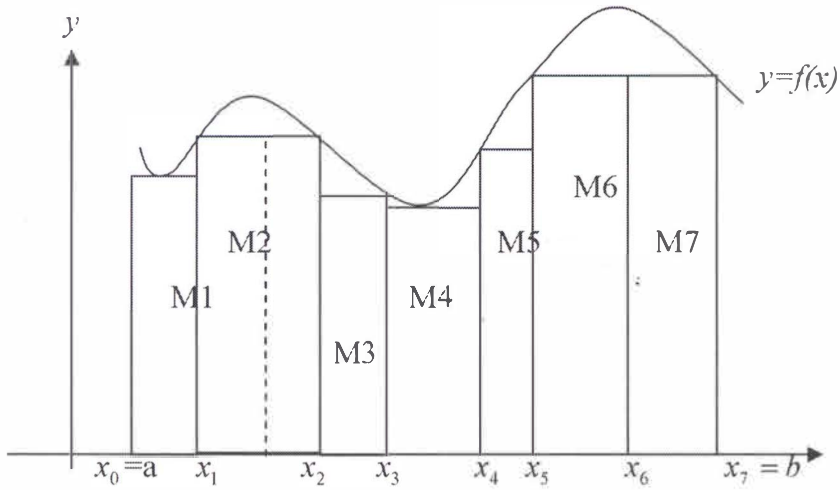


Figure 8: L(P, f)

If $f \geq 0$, Figure 7 showed that the upper sum for a nonnegative continuous function f. $U(P, f)$ represents the circumscribed rectangular approximation to the area under the graph of f. The lower sums were representing by the figure 8 where the inscribed rectangular approximation to the area under the graph of f. We will add up the area from each rectangle created from each piece. For each piece, we have the upper and lower estimated of the area under $f(x)$, that is,

$$M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]$$

and

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i].$$

Also defined

$$U(P, f) = \sum_{i=1}^{n} M_i(f(x))(x_i - x_{i-1})$$

and
$$L(P, f) = \sum_{i=1}^{n} m_i(f(x))(x_i - x_{i-1}).$$

Then we have our Riemann sum integral. Now assumed the function f bounded on the arbitrary partition P $[a, b]$, so there is a real number M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. For

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b - a)$$

and

$$L(P,f)=\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} m(x_i - x_{i-1}) = m(b-a).$$

Then,

$$U(P,f)=M(b-a) \text{ and } L(P,f)=m(b-a),$$

for the similar condition as above, w is the width of the rectangle, since $m \le m_i \le M_i \le M$, for i=1,...., n. It follows that,

$$mw_i \le m_i w_i \le M_i w_i \le Mw_i$$

for each i. If we sum the n inequalities for i=1....., n, we obtain

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a).$$

Above inequality was showed established that for a fixes bounded function, f, the collection of all upper sums as well as the collection of all lower sums over f is bounded below by $m(b\text{-}a)$ and bounded above by $M(b\text{-}a)$.

Definition 3.1.3: Let f be a bounded real-valued function on the closed and bounded interval [a, b]. The upper and lower integrals of f, is denoted by $\overline{\int_a^b} f(x)dx = \inf U(P,f)$ and $\underline{\int_a^b} f(x)dx = \sup L(P,f)$ respectively, where P is a partition of [a, b]. Since the set of lower sums $L(P, f)$ for all possible partitions is bounded above by $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$, the lower integral exists. Similarly, the set of all upper sums $U(P,f)$ is bounded below for every partition , the upper integral exists. Then from the inequality $L(P,f) \le U(P,f)$, we have

$$\underline{\int_a^b} f < \overline{\int_a^b} f.$$

Definition 3.1.4: If f is a bounded real value-function on a closed and bounded interval [a, b], then f is said to be Riemann integrable on [a, b] provided $\overline{\int_a^b} f = \underline{\int_a^b} f$.

The common value is denoted by $\int_a^b f$ and is called the set of Riemann integrable of f over [a, b]. We denoted by R[a ,b] the set of Riemann integrable functions on [a, b]. If $f \in R[a,b]$, then defined that $\int_b^a f = -\int_a^b f$. If $f:[a,b] \to R$ satisfies $m \le f(t) \le M$ for all $t \in [a,b]$, then $m(b-a) \leq \int_{-a}^{b} f < \int_{a}^{-b} f \leq M(b-a)$ If $f \in R[a,b]$, then

$$m(b-a) \leq \int_{a}^{b} f \leq M(b-a).$$

In particular, if $f(x) \geq 0$ for all $x \in [a,b]$ and $f \in R[a,b]$ is nonnegative, then the quantity $\int_{a}^{b} f$ represents the area of the region bounded above by the graph $y = f(x)$, below by the x-axis, and by the lines $x=a$, and $x=b$.

Definition 3.1.5: A partition P^* of $[a, b]$ is a refinement of P if $P \subset P^*$. A refinement of a given partition P is obtained by adding additional points to P. If P_1 and P_2 are two partitions of $[a, b]$, then $P_1 \cup P_2$ is a refinement of both P_1 and P_2.

Remark: By the words, any refinement of the given partition increases the lower sums and but reduces the upper sums.

i. Since P^* is a finite set which contain P, then P^* can be obtained from P by adding in a finite number of points, one at a time. With showing in general, adding a single point into a partition P causes the lower Riemann sum to increase, then clearly adding in finitely many points one after the other will also increase the Riemann sum from its original value. Suppose that P^* is obtained from P by adding in just one more point z. If $P = \{x_0, x_1, ..., x_n\}$ then there has to be a n between 1 and k such that $x_{n-1} < z < x_n$.

Figure 9: Refinement of partition P

Let,

$$m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

$$s_1 = \inf\{f(x) : x \in [x_{j-1}, z]\},$$
$$s_2 = \inf\{f(x) : x \in [z, x_j]\},$$
$$r_1 = \sup\{f(x) : x \in [x_{j-1}, z]\},$$

and $$r_2 = \sup\{f(x) : x \in [z, x_j]\}.$$

Now,

$$m_i(f) \leq \min\{s_1, s_2\} \quad \text{and} \quad M_i(f) \geq \max\{r_1, r_2\}.$$

Then,

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{j-1} m_i(f)(x_i - x_{i-1}) + m_j(f)(x_j - x_{j-1}) + \sum_{i=j+1}^{n} m_i(f)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{j-1} m_i(f)(x_i - x_{i-1}) + s_i(z - x_{j-1}) + s_2(x_j - z) + \sum_{i=j+1}^{n} m_i(f)(x_i - x_{i-1}) \\
&= L(f, P^*),
\end{aligned}$$

and

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{j-1} M_i(f)(x_i - x_{i-1}) + M_j(f)(x_j - x_{j-1}) + \sum_{i=j+1}^{n} M_i(f)(x_i - x_{i-1}) \\
&\geq \sum_{i=1}^{j-1} M_i(f)(x_i - x_{i-1}) + r_1(z - x_{j-1}) + r_2(x_j - z) + \sum_{i=j+1}^{n} M_j(f)(x_i - x_{i-1}) \\
&= U(f, P^*),
\end{aligned}$$

thus,

$$L(f,P) \leq L(f,P^*)$$

and

$$U(f,P^*) \leq U(f,P).$$

ii. For any partition P of $[a, b]$, $m_i(f) \leq M_i(f)$; hence $L(f,P) \leq U(f,P)$. Now, if P and P^* are any partition of $[a, b]$, then $P \cup P^*$ is also a partition of $[a, b]$, which is refined both P and P^*, then we have

$$U(f,P) \geq U(f, P \cup P^*)$$
$$L(f, P \cup P^*) \geq L(f,P)$$

and (i) asserts that

$$U(f, P \cup P') \geq L(f, P \cup P'),$$

therefore, we will have

$$U(f,P) \geq U(f, P \cup P^*) \geq L(f, P \cup P^*) \geq L(f,P)$$

and

$$U(f,P) \geq L(f,P). \square$$

Theorem 3.1.6: Let f be bounded function on $[a, b]$. Then every upper sum for f is greater than or equal to every lower sum for f. So, if P_1 and P_2 are any two partitions on $[a, b]$, then $U(f,P_1) \geq L(f,P_2)$.

Proof: Since $P_1 \cup P_2$ refines both P_1 and P_2, we have

$$U(f,P) \geq U(f,P_1 \cup P_2)$$

$$L(f,P_1 \cup P_2) \geq L(f,P_2).$$

From the Definition 3.1.2 have asserts that $U(f,P_1 \cup P_2) \geq L(f,P_1 \cup P_2)$. Therefore

$$U(f,P_1) \geq L(f,P_2). \square$$

Theorem 3.1.7: Let f is a bounded function on the closed and bounded interval $[a, b]$. Then f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a subdivision P of $[a, b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof: Let

$$\overline{\int_a^b} f(x)dx \leq U(f,P) \qquad \text{and} \qquad \underline{\int_a^b} f(x)dx \geq L(f,P).$$

Hence, we will get

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then

$$\overline{\int_a^b} f(x)dx \leq \underline{\int_a^b} f(x)dx$$

such that $U[P,f] - L[P,f] < \varepsilon$. We have $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$, hence we have

$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$, so that f is Riemann integrable in $[a, b]$. Then,

$$\overline{\int_a^b} f = \sup U(f,P) = \inf L(f,P) = \underline{\int_a^b} f .$$

Given $\varepsilon > 0$ from the definition of supremun, we can choose for a partition P_1 and P_2 respectively such that

$$U(f,P_1) < \overline{\int_a^b} f + \frac{\varepsilon}{2}, \quad \text{and} \quad L(f,P_2) < \underline{\int_a^b} f - \frac{\varepsilon}{2}.$$

Using the fact that f is Riemann integrable, we get

$$L(f,P_2) + \frac{\varepsilon}{2} > U(f,P_1) - \frac{\varepsilon}{2}.$$

Now considering the common partition of P_1 and P_2

$$L(f,P_1 \cup P_2) + \frac{\varepsilon}{2} > U(f,P_1 \cup P_2) - \frac{\varepsilon}{2}$$

and considering $P_1 \cup P_2$ as single partition P, we get

$$U(f,P) - L(f,P) < \varepsilon . \square$$

Discussion: This is the theorem we will often apply to check the integrability of a function. The tool for obtaining the desired partition will be the clever manipulation of the norm. Specifically, we will make the norm small. In general, if the ε small, the norm of P will have to be small as well to guarantee that the difference $U(f,P) - L(f,P) < \varepsilon$.

Theorem 3.1.8: Let f be a real-valued functions on $[a, b]$.

i. If f is monotone on $[a, b]$, then f is Riemann integrable on $[a, b]$.

ii. If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Proof: i) If f is constant on $[a, b]$, then f is Riemann-integrable on $[a, b]$. We assume that f is monotonic increasing on $[a, b]$ and $f(a) = f(b)$. The case for f decreasing is similar. There exists a partition P on $[a, b]$ for which $U(f,P) - L(f,P) < \varepsilon$. Choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that $\| P \| = \dfrac{\varepsilon}{f(b) - f(a)}$, since f is increasing on $[a, b]$, we have

$$M_i(f) = f(x_i) \quad \text{and} \quad m_i(f) = f(x_{i-1}), \quad i = 1, 2, \ldots, n.$$

Hence, $$U(f,P)-L(f,P)=\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})][x_i-x_{i-1}]$$

$$<\frac{\varepsilon}{f(b)-f(a)}\sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]\backslash$$

$$=\frac{\varepsilon}{f(b)-f(a)}[f(x_n)-f(x_0)]$$

$$=\varepsilon.$$

ii) Suppose f is continuous on $[a, b]$ and let $\varepsilon > 0$ be given. A partition P for $[a, b]$ exists, such that

$$U(f,P)-L(f,P)<\varepsilon.$$

By the uniformly continuity of f on $[a, b]$, there is a $\delta > 0$ such that

$$|f(x)-f(y)|<\frac{\varepsilon}{b-a},$$

whenever $x, y \in [a,b]$ with $|x-y|<\delta$. Let P be any partition of $[a, b]$ with $\|P\|<\delta$. By the property of continuous function on the closed interval $[x_{i-1}, x_i]$, there exist points $t_i, s_i \in [x_{i-1}, x_i]$ such that

$$M_i(f)=f(t_i) \quad \text{and} \quad m_i(f)=f(s_i), \qquad i=1, \ldots, n.$$

Now,

$$|x_i-x_{i-1}|<\delta, \qquad |t_i-s_i|<\delta,$$

and hence

$$M_i-m_i=|f(t_i)-f(s_i)|<\frac{\varepsilon}{b-a}$$

Then, we have

$$U(f,P)-L(f,P)=\sum_{i=1}^{n}[(f(t_i)-f(s_i)](x_i-x_{i-1})$$

$$=\sum_{i=1}^{n}\frac{\varepsilon}{b-a}(x_i-x_{i-1})$$

$$=\frac{\varepsilon}{b-a}(b-a)$$

$$=\varepsilon. \ \square$$

Discussion: The monotonicity of the function f guarantees that the maximum and the minimum values occur at the two ends points of its subintervals. So, if we choose a partition with all subintervals having an equal length, d, then since $\sum(M_i - m_i) = f(b) - f(a)$, its turn out that $U(f,P) - L(f,P) = d \times [f(b) - f(a)]$. This justifies the quantity d to be less than $\dfrac{\varepsilon}{f(b)-f(a)}$. For the Theorem 3.1.8(ii), a continuous function on a closed interval $[a, b]$, is uniformly continuous. $|f(t_i) - f(s_i)|$ has the extraordinary property that it can be made as small as we please provided t_i and s_i are sufficiently close. The required closeness can be ensured by the single step of making the norm of P sufficiently small.

Theorem 3.1.9: Let f be a bounded Riemann integrable function on $[a, b]$ with Range $f \subset [c,d]$. If the φ is continuous on $[c, d]$, then the composition $\varphi \circ f$ is Riemann integrable on $[a, b]$.

Proof: Let $\varepsilon > 0$, then we shall prove the existence of a partition $P \in P(a,b)$ such that,

$$U(\varphi \circ f) - L(\varphi \circ f) < \varepsilon.$$

Since φ is continuous on the compact interval $[c, d]$, it is bounded and uniformly continuous. Consequently there is a real constant K such that

$$|\varphi(t)| \leq K \qquad \text{for all } t \in [c,d],$$

and if we set $\varepsilon' = \dfrac{\varepsilon}{2K + (b-a)}$, we know that there is a $\delta > 0$ such that

$$s,t \in [c,d], \qquad |t - s| < \delta \Rightarrow |\varphi(t) - \varphi(s)| < \varepsilon'.$$

On the other hand, since $f \in R(a,b)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$U(f,P) - L(f,P) < \varepsilon'\delta. \qquad (3.1.1)$$

Let,

$$\begin{aligned}
m_i &= \inf\{f(x) : x \in [x_i, x_{i+1}]\}, \\
m_i^* &= \inf\{\varphi(f(x)) : x \in [x_i, x_{i+1}]\}, \\
M_i &= \sup\{f(x) : x \in [x_{i+1}, x_i]\}, \\
M_i^* &= \sup\{\varphi(f(x)) : x \in [x_i, x_{i+1}]\}.
\end{aligned}$$

Then we have

$$U(\varphi \circ f) - L(\varphi \circ f) = \sum_{i=0}^{n-1} (M_i^* - m_i^*)(x_{i+1} - x_i)$$

$$= \sum_{i \in J_1} (M_i^* - m_i^*)(x_{i+1} - x_i) + \sum_{i \in J_2} (M_i^* - m_i^*)(x_{i+1} - x_i),$$

where

$$J_1 = \{i \in \{0,1,\ldots,n-1\} : M_i - m_i < \delta\}$$
$$J_2 = \{i \in \{0,1,\ldots,n-1\} : M_i - m_i \geq \delta\},$$

therefore,

$$|\varphi(f(x)) - \varphi(f(y))| < \varepsilon' \quad \text{for all } x, y \in [x_i, x_{i+1}],$$

which implies that $M_i^* - m_i^* \leq \varepsilon'$. Therefore,

$$\sum_{i \in J_1} (M_i^* - m_i^*)(x_{i+1} - x_i) \leq \sum_{i \in J_1} \varepsilon'(x_{i+1} - x_i) \leq \varepsilon'(b-a).$$

Then, from equation (3.1.1),

$$\varepsilon'\delta > \sum_{i=0}^{n-1} (M_i - m_i)(x_{i+1} - x_i)$$
$$\geq \sum_{i \in J_2} (M_i - m_i)(x_{i+1} - x_i)$$
$$\geq \delta \sum_{i \in J_2} (x_{i+1} - x_i).$$

Hence,

$$\sum_{i \in J_2} (x_{i+1} - x_i) < \varepsilon'.$$

Since $M_i^* - m_i^* \leq 2K$, we must have

$$\sum_{i \in J_2} (M_i^* - m_i^*)(x_{i+1} - x_i) < 2K\varepsilon'.$$

Then this inequalities yield

$$U(\varphi \circ f) - L(\varphi \circ f) < \varepsilon'(b-a) + 2K\varepsilon' = \varepsilon. \ \square$$

Discussion: For the Theorem 3.1.9, the composition of continuous functions is continuous and the composition of differentiable functions is defferentiables, one might conjecture that, if $f:[a,b] \to [c,d]$ and $g:[c,d] \to R$ are such that $f \in R(x)$ on $[a, b]$ and $g \in R(x)$ on $[c, d]$, then $g \circ f \in R(x)$ on $[a, b]$.

Example 3.1.10: Consider the function $f(x) = x^2, x \in [0,1]$. For $n \in N$, let P_n be the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$. Since f is increasing on [0, 1], its infimum and supremum on each interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ are attained at the left and right endpoint respectively, with

$$m_i = \frac{(i-1)^2}{n^2} \text{ and } M_i = \frac{i^2}{n^2}.$$

Since $\Delta x_i = 1/n$ for all i,

$$L(P_n, f) = \frac{1}{n^3}[1^2 + 2^2 + ... + (n-1)^2],$$

and

$$U(P_n, f) = \frac{1}{n^3}[1^2 + 2^2 + ... + n^2].$$

Then, $$L(P_n, f) = \frac{1}{6}(1-\frac{1}{n})(2-\frac{1}{n})$$

and $$U(P_n, f) = \frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n}).$$

Thus $\sup_n L(P_n, f) = \frac{1}{3}$ and $U(P_n, f) = \frac{1}{3}$. Since the collection $\{P_n : n \in N\}$ is a subset of the set of all partition of [0, 1],

$$\frac{1}{3} = \sup_n L(P_n, f) \leq \sup_P L(P_n, f) = \int_0^1 x^2 dx$$

and

$$\frac{1}{3} = \inf_n U(P_n, f) \leq \inf_P U(P_n, f) = \int_0^1 x^2 dx.$$

Therefore $f(x) = x^2$ is integrable on [0,1] with $\int_0^1 x^2 dx = \frac{1}{3}$.

Example 3.1.11: By consider the Theorem 3.1.9, define $f:[1,0] \to R$ by $f(x) = 0$ if x is irrational, and $f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$ with p and q relatively prime nonnegative integers, $q \neq 0$. It has already been shown that $f \in R(x)$ on [0, 1]. Define $g:[0,1] \to R$ by $g(x) = 1$ if $0 < x \leq 1$ and $g(0) = 0$. Let $f = g \circ f$. Then if x is irrational, $h(x) = 0$ and if x is rational, $h(x) = 1$. Then h is not integrable on [0, 1].

3.2 Properties of the Riemann Inetegral

The properties express how the integration behaves. Riemann integral keeps all of the basic properties of the integral of the continuous functions. We will derive the basic properties of the Riemann integral. We consider the closed and bounded interval $[a, b]$, $a < b$ in R, and $R[a, b]$ denotes the set of Riemann integrables functions on $[a, b]$. Our first result proves that the integral is additive.

Theorem 3.2.1: Let $f, g \in R[a,b]$ are integrable functions, then $f+g \in R[a,b]$, and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof: First, consider any partition $P: a = x_0, x_1, \ldots, x_n = b$ of $[a, b]$ and let

$$m_i(f) = \inf\{f(x) : x_{i-1} \le x \le x_i\},$$

with corresponding for $m_i(g), and \ m_i(f+g)$. This implies that

$$L(f+g, P) \ge L(f, P) + L(g, P), \tag{3.2.1}$$

$$U(f+g, P) \le U(f, P) + U(g, P), \tag{3.2.2}$$

for any partition P. Now, we consider any $\varepsilon > 0$. Since f and g are integrable, there exists partition P_f and P_g such that

$$\int_a^b f - \frac{\varepsilon}{2} < L(f, P_f) \le U(f, P_f) < \int_a^b f + \frac{\varepsilon}{2}$$

$$\int_a^b g - \frac{\varepsilon}{2} < L(f, P_g) \le U(f, P_g) < \int_a^b g + \frac{\varepsilon}{2}.$$

The partition P obtained by taking all the points of P_f and P_g together is a refinement of both and therefore both the inequalities displayed above hold if P_f and P_g are both replaces by P. By adding two inequalities obtained by making the replacement, we get

$$\int_a^b f + \int_a^b g - \varepsilon < L(f, P) + L(g, P) \le U(f, P) + U(g, P) \le + \int_a^b f + \int_a^b g + \varepsilon.$$

By combining the equation (3.2.1) and (3.2.2), since a similar statement is valid for lower sums, we have,

$$\int_a^b f+\int_a^b g-\varepsilon < L(f+g,P) \quad \text{and} \quad U(f+g,P) < \int_a^b f+\int_a^b g+\varepsilon .$$

Since such a partition P has been shown to exist for any positive ε, it follows that

$$\int_a^b f+\int_a^b g \le \int_a^b f+g \quad \text{and} \quad \overline{\int_a^b} (f+g) \le \int_a^b f+\int_a^b g .$$

Since the lower integral never exceeds the upper Riemann integral, then f+g are integrable and further that

$$\underline{\int_a^b} (f+g) = \overline{\int_a^b} (f+g) = \int_a^b f+\int_a^b g.$$

is also integrable. □

Theorem 3.2.2: If f is integrable on $[a, b]$ and c is any constant, then cf is integrable on $[a, b]$ and

$$\int_a^b cf = c\int_a^b f .$$

Proof: First consider any partition $P: a = x_0, x_1, ..., x_n = b$ of $[a, b]$ and let

$$m_i(f) = \inf\{f(x) : x_{i-1} \le x \le x_i\} \quad , \quad M_i(f) = \sup\{f(x) : x_{i-1} \le x \le x_i\}$$

which corresponding meaning for

$$m_i(cf) = cm_i(f) \quad \text{and} \quad M_i(cf) = cM_i(f) \tag{3.2.3}$$

whenever $c > 0$. Therefore $L(cf, P)=cL(f, P)$ and $U(cf, P)=cU(f, P)$ for any partition P. Hence, whenever f is any bounded function,

$$\underline{\int_a^b} cf = c\underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} cf = c\overline{\int_a^b} f . \tag{3.2.4}$$

This show that, when f is integrable and c >0, the function cf is also integrable and $\int_a^b cf = c\int_a^b f$. When c <0, instead of equation (3.2.3) and (3.2.4), we will then have

$$m_i(cf) = cM_i(f) \quad \text{and} \quad M_i(cf) = cm_i(f)$$

and

$$\underline{\int_a^b} cf = c\overline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} cf = c\underline{\int_a^b} f .$$

This show that, when f is integrable and $c < 0$, the function cf also integrable and

$$\int_a^b cf = c\int_a^b f . \square$$

Theorem 3.2.3 (Linear property): If f and g are both integrable on $[a, b]$, then the linear combination is also integrable. For, c_1 and c_2 any real numbers, $c_1 f + c_2 g \in R[a,b]$, then,

$$\int_a^b (c_1 f + c_2 g)dx = c_1 \int_a^b fdx + c_2 \int_a^b gdx) .$$

Proof: Choose $\varepsilon > 0$. There is $\varepsilon' > 0$ such that

$$(| c_1 | + | c_2 |)\varepsilon' \leq \varepsilon .$$

There are partitions of P regardless to any refinement P^*, $\left|S(f_i, P_i^*) - \int_a^b f_i dx\right| \leq \varepsilon'$. Let $P = P_1 \cup P_2$, if P^* is any refinement of P, then P^* is a refinement of P_1 and a refinement of P_2, hence,

$$\left|S(c_1 f_1 + c_2 f_2, P^*) - \left[c_1 \int_a^b f_1 dx + c_2 \int_a^b f_2 dx\right]\right| = \left|c_1 S(f_1, P^*) + c_2 S(f_2, P^*) - c_1 \int_a^b f_1 dx - c_2 \int_a^b f_2 dx\right|$$

$$\leq | c_1 | \left|S(f_1, P^*) - \int_a^b f_1 dx\right| + | c_2 | \left|S(f_2, P^*) - \int_a^b f_2 dx\right|$$

$$\leq | c_1 | \varepsilon' + | c_2 | \varepsilon' \leq \varepsilon.$$

Hence, $c_1 f + c_2 g \in R(x)$ on $[a, b]$ and

$$\int_a^b (c_1 f + c_2 g)dx = c_1 \int_a^b fdx + c_2 \int_a^b gdx) . \square$$

Theorem 3.2.4: Let $f \in R[a,b]$, if $f(x) \geq 0$ almost everywhere on $[a, b]$, then

$$\int_a^b f(x)dx \geq 0 .$$

Proof: From the Definition 3.1.2, we have the following situation that is

$$m_i(f)(b-a) \le \int_a^b f \le M_i(f)(b-a).$$

If $f(x) > 0$ for every $x \in [a,b]$, then $m_i(f) > 0$. By using the above inequalities, the $\int_a^b f \ge 0$ is proved. □

Theorem 3.2.5: If a function f continuous on $[a, b]$, $f(x) \ge 0$ for $a \le x \le b$, and if $\int_a^b f(x)dx = 0$, then f is identically zero on $[a, b]$.

Proof: If f is not identically zero on $[a, b]$, there exists a point c in $[a, b]$ such that $f(c) > 0$. Now f is continuous function in the bounded and closed interval $[a, b]$ and $f(x) \ge 0$. Since $f(c) > 0$ for $c \in [a,b]$, $\int_a^b f > 0$ are contradicts with the hypothesis. Hence f is identically zero on $[a, b]$. □

Theorem 3.2.6 (Monotone property)**:** If $f \in R[a,b]$ and $g \in R[a,b]$ and $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

Proof: Since $g(x) - f(x) \ge 0$, every lower sum of g-f over any partition of $[a, b]$ is nonnegative. Therefore,

$$\int_{\underline{a}}^{b} (g(x) - f(x))dx \ge 0.$$

Hence,

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx$$

$$= \int_{\underline{a}}^{b} (g(x) - f(x))dx \ge 0$$

which already prove that

$$\int_a^b f(x)dx \le \int_a^b g(x)dx . \square$$

Theorem 3.2.7 (Absolute property): If $f \in R[a,b]$ and is integrable, then so is $|f|$ and

$$|\int_a^b f| \ge \int_a^b |f| .$$

Proof: Since $|f|$ is continuous at every point where f is continuous, $|f| \in R[a,b]$. Since $f(x) \le |f(x)| = |f|(x)$ for every $x \in [a,b]$, and from the Theorem 3.2.4 above, we get the form

$$\int_a^b f \le \int_a^b |f| . \qquad (3.2.5)$$

Since $-f(x) < |f|(x)$ for all $x \in [a,b]$, we have again using the theorem 3.2.6 above,

$$\int_a^b f \ge -\int_a^b |f| . \qquad (3.2.6)$$

From equation (3.2.5) and (3.2.6), since both of the f and $|f|$ is integrable, then will prove that

$$|\int_a^b f| = \int_a^b |f| . \square$$

Theorem 3.2.8: If $f \in R[a,b], then\, |f| \in R[a,b]$.

Proof: Since f is bounded in $[a, b]$, $|f(x)| \le i$ for every $x \in [a,b]$ so that $|f|$ is bounded. Let $\varepsilon > 0$ be given and let $P : a = \{x_0, x_1, \ldots, x_n\} = b$ be a partition of $[a, b]$ and let $x, y \in P$. Then we have the following

$$\left\|| f(x) - |f(y)|\right\| \le |f(x) - f(y)| \le M_i(f) - m_i(f) .$$

As x, y vary over $[x_{i-1}, x_i]$, then

$$M_i(|f|) - m_i(|f|) \le M_i(f) - m_i(f) .$$

This implies that

$$U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P) . \qquad (3.2.7)$$

Since $f \in R[a,b]$, from

$$U(f,P) - L(f,P) < \varepsilon \text{ for every } \varepsilon > 0. \tag{3.2.8}$$

Using both equation (3.2.7) and (3.2.8), we will get

$$U(|f|,P) - L(|f|,P) < \varepsilon.$$

Hence,

$$|f| \in R[a,b]. \square$$

Discussion: But converse of above theorem are not true. It can be shown by following example:

Let $f \in R[a,b]$ defined on $[a, b]$ by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational,} \\ -1 & \text{when } x \text{ is irrational,} \end{cases}$$

for any partition of $[a, b]$. Then we can check easily that

$$\overline{\int_a^b} f = (b-a) \quad \text{and} \quad \overrightarrow{\int_a^b} f = -(b-a).$$

This implies that f is not Riemann integrable in $[a, b]$. but $|f(x)|=1$ for every $x \in [a,b]$. Hence $|f|$ is Riemann integrable and its value equals to $(b - a)$.

Theorem 3.2.9: If f is integrable on $[a, b]$ and $c \in [a,b]$, then f is integrable on $[a, c]$ and $[c, b]$ and further

$$\int_a^b f = \int_c^b f + \int_a^b f.$$

Proof: If $f \in R[a,c]$ and $f \in R[c,b]$, if P is any partition of $[a, c]$ and Q is any partition of $[c, b]$, then $P \cup P^*$ is a partition of $[a, b]$ whose component intervals are those of P together with those P^*. Hence, we have

$$L(f,P) + L(f,P^*) = L(f, P \cup P^*) \leq \underline{\int_a^b} f$$

and so,

$$L(f,P) + L(f,P^*) \leq \underline{\int_a^b} f.$$

By taking the least upper bound on the left over all P, keeping P^* fixed, we obtain

$$\int_a^c f + L(f, P^*) \le \int_a^b f.$$

Now taking least upper bound over all P*, we get

$$\int_a^c f dx + \int_c^b f \le \int_a^b f. \qquad (3.2.9)$$

By using similar argument by considering the upper sums, we get the reverse inequality,

$$\int_a^c f + \int_c^b f \ge \int_a^b f. \qquad (3.2.10)$$

From equation (3.2.9) and (3.2.10), we will obtain

$$\int_a^b f = \int_a^c f + \int_a^b f. \ \square$$

Discussion: The particular above theorem has an important interpretation for nonnegative functions. If we split the interval over which we are integrating into two parts, the value of the integral over the whole will be the sum of the two integral over the subintervals. This amounts to dividing the region whose area must be found into two separate parts and observing that the total area is the sum of the areas of the separate portions.

Theorem 3.2.10: If $f \in R[a,b]$,

i. $f \in R[c,d]$ for every subinterval $[c,d] \subset [a,b]$.

ii. $f^2 \in R[a,b]$.

iii. $f \cdot g \in R[a,b]$ whenever $g \in R[a,b]$.

iv. If $f, g \in R[a,b]$, then $f / g \in R[a,b]$, where g is bounded away from zero.

v. If f and g are bounded functions having the same discontinuities on $[a, b]$, then $f \in R[a,b]$ if and only if $g \in R[a,b]$

vi. Let $g \in R[a,b]$ and assume that $m \le g(x) \le M$ for all $x \in [a,b]$. If f is continuous on $[m, M]$, then the composite function defined by $h(x) = f[g(x)]$ is Riemann integrable on $[a, b]$.

Proof: i) Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a, b]$ such that

$$U(f,P)[a,b] - L(f,P)[a,b] < \varepsilon.$$

Let $P^* = P \cup \{c,d\}$. The P* is a refinement of $[a, b]$ then

$$U(f,P^*)[a,b] \le U(f,P)[a,b]$$

and

$$L(f,P^*)[a,b] \ge L(f,P)[a,b].$$

Now, let $Q = P^* \cap [c,d]$. Then Q is obtained by restricting P^* to $[c, d]$. Hence we have the inequality,

$$U(f,Q)[c,d] - L(f,Q)[c,d] \le U(f,P^*)[a,b] - L(f,P^*)[a,b]. \qquad (3.2.11)$$

Because of the left hand side has fewer terms which are all non-negative than the right hand side. Since $f \in R[a,b]$, we get

$$U(f,P^*)[a,b] - L(f,P^*)[a,b] < \varepsilon. \qquad (3.2.12)$$

Using equation (3.2.12) in equation (3.2.11), we get that

$$U(f,Q)[c,d] - L(f,Q)[c,d] < \varepsilon.$$

Therefore, we get $f \in R[c,d]$.

ii) Let $\varepsilon > 0$ be given and than there exists a partition P of $[a, b]$ such that

$$U(f,P)[a,b] - L(f,P)[a,b] < \varepsilon.$$

Since, we know that $M_i(f^2) = M_i(|f|)^2$ and $m_i(f^2) = m_i(|f|)^2$.

$$U(f^2,P) - L(f^2,P) = \sum_{i=1}^{n}[M_i(f^2) - m_i(f^2)]\,[x_{i-1},x_i]$$

$$= \sum_{i=1}^{n}[M_i(|f|)^2 - m_i(|f|)^2][x_{i-1},x_i]$$

$$= \sum_{i=1}^{n}\{M_i(f) + m_i(f)\}\{M_i(|f|) - m_i(|f|)\}[x_{i-1},x_i]$$

$$\le 2\lambda\sum_{i=1}^{n}\{M_i(|f|) - m_i(|f|)\}[x_{i-1},x_i].$$

Where λ is an upper bound of f in $[a, b]$. Therefore, we have,

$$U(f^2,P) - L(f^2,P) < 2\lambda[U(|f|,P) - L(|f|,P)]$$

by

$$U(|f|,P) - L(|f|,P) < \frac{\varepsilon}{2\lambda}.$$

Hence,

$$U(f^2, P) - L(f^2, P) < \varepsilon$$

and therefore , $f^2 \in R[a,b]$.

iii) Firstly, we shall show that the square of a Riemann integrable function is also Riemann integrable, such that

$$M_i(f^2) = M_i(|f|)^2 \quad \text{and} \quad m_i(f^2) = m_i(|f|)^2$$

Thus,

$$U(f^2, P) - L(f^2, P) = \sum_{i=1}^{n} [M_i(f^2) - m_i(f^2)][x_{i-1}, x_i]$$

$$= \sum_{i=1}^{n} \{M_i(f) + m_i(f)\}\{M_i(|f|) - m_i(|f|)\}[x_{i-1}, x_i]$$
$$\leq 2\lambda[U(|f|, P) - L(|f|, P)].$$

We have shown that the square of any Riemann integrable function is Riemann integrable, now, *f* and *g* are Riemann inetgarable,

f(x), *g(x)* and *f(x)+g(x)*

are all Riemann integrable, thus,

$$(f(x) + g(x))^2 - f(x)^2 - g(x)^2 = 2f(x)g(x)$$

is Riemann integrable, and so *f(x)g(x)* is integrable.

iv) Since $g(x) \neq 0$ for any $x \in [a,b]$, applying $f \cdot \frac{1}{g} \in R[a,b]$, provided

$\frac{1}{g} \in R[a,b]$ whenever $g \in R[a,b]$ under the given condition. Hence we shall prove

that $\frac{1}{g} \in R[a,b]$, whenever $g \in R[a,b]$ and g is bounded away from zero. We have,

$|g(x)| > i$ for every $x \in [a,b]$. Let P be a partition on $[a, b]$, $\alpha, \beta \in [x_{i-1}, x_i]$.

$$\left|\frac{1}{g(\alpha)} - \frac{1}{g(\beta)}\right| = \left|\frac{g(\beta) - g(\alpha)}{g\alpha)g(\beta)}\right| < \frac{1}{i^2}|g(\beta) - g(\alpha)|.$$

Then,

$$M_i\left(\frac{1}{g}\right) - m_i\left(\frac{1}{g}\right) < \frac{1}{i^2}[M_i(g) - m_i(g)],$$

this implies that

$$U\left(\frac{1}{g},P\right)-L\left(\frac{1}{g},P\right)<\frac{1}{i^2}[U(g,P)-L(g,P)].$$

Since $g\in R[a,b]$, given $\varepsilon>0$, there exists a partition P such that

$$U(g,P)-L(g,P)<i^2\varepsilon .$$

Then we will get

$$U\left(\frac{1}{g},P\right)-L\left(\frac{1}{g},P\right)<\varepsilon . \square$$

3.3 The Fundamental Theorem of Integral Calculus

At this stage in our development, we have proven several theorems for testing whether a given function is integrable on an interval $[a, b]$. But this fact does not give us a method for finding the value of the integral. It would be convenient to have a procedure to compute easily the actual value of an integral. Thus the fundamental Theorem of calculus describes an important connection between integrals and derivatives as well as to compute the integrals. The connection between the Riemann integral and antiderivative are often called "indefinite" and "definite".

We have already seen that the definite integral of a positive function can be interpreted as the area under the graph of the function. The definite integral is given by the a sum which is

$$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x .$$

When the function f was positive, we could interpret each term $f(x_i)\Delta x$ as the area of a very thin rectangle. This means when we perform the sum $\sum_{i=0}^{n-1} f(x_i)\Delta x$, we are actually computing the additive inverse of the area of all the rectangles and so the area A is given by

$$A=-\int_a^b f(x)dx \qquad \text{or} \qquad \int_a^b f(x)dx=-A .$$

The definite integral measures the additive inverse of the area between the graph and the x-axis. Now, when the function f has some regions where it is positive and others where it is negative, we may compute the definite integral by integrating over the regions where the function is positive and add that to the integral over the region where the function is negative. This leads to the observation that

$$\int_a^b f(x)dx = A_1 - A_2 .$$

Where A_1 is the area bounded by the region where the function is positive and A_2 is the area bounded by the region where the function is negative. So in this most general case, the definite integral can still be thought of as measuring area, but it does by measuring some area as negative and others as positive. With this interpretation, we can convey the following Fundamental Theorem of Calculus.

Theorem 3.3.1 (The First Fundamental Theorem of Calculus): Suppose that $f \in R[a,b]$ is continuous on the closed bounded interval $[a, \mathrm{b}]$ and if

$$F(x) = \int_a^x f(t)dt ,$$

then,

$$F'(x) = f(x)$$

for all $a \le x \le b$.

Proof: If $f(x)$ is continuous function on $F(x) = \int_a^x f(t)dt$, then

$$F'(x) = f(x).$$

For any fixed $x \in [a,b]$, choose $h \neq 0$ and $x + h \in [a,b]$. Then, we have the following

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt. \end{aligned}$$

Notice that as h becomes small, the definite integral can be thought of as the area of a very thin strip. In particular, the area of this trip may be approximated by the area of a rectangle. This means that

$$F(x+h)-F(x)=\int_{x}^{x+h} f(t)dt \approx f(x)h$$

or

$$\frac{F(x+h)-F(x)}{h}=f(x)h.$$

Now as h becomes very small, the approximation becomes even better and so we have

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)h.$$

Thus verifying the relationship we have been expect. This completes the argument which justifies the relationship

$$F'(x)=f(x). \square$$

Theorem 3.3.2 (Second Fundamental Theorem of Calculus): Let $f\in R[a,b]$ be integrable and $F\in R[a,b]$ is an any antiderivative of f on (a, b) which is continuous function such that

$$F'(x)=f(x), \quad \forall x\in[a,b].$$

Then

$$\int_{a}^{b} f = F(b)-F(a), \quad \forall x\in[a,b].$$

Proof: Suppose f is integrable on $[a, b]$ and F is an antiderivative of f on (a, b) which is continuous on $[a, b]$. In particular, $F'(x) = f(x)$ for all x in (a, b). Let $P=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a, b]$, and let $\Delta x_i = x_i - x_{i-1}$, $i=1,2,\dots,n$. Now,

$$F(b)-F(a)=F(x_n)-F(x_0)$$

$$=F(x_n)+(F(x_{n-1})-F(x_{n-1}))+(F(x_{n-2})-F(x_{n-2}))+\dots+(F(x_1)-F(x_1))-F(x_0)$$
$$=(F(x_n)-(F(x_{n-1}))-F(x_{n-1}))+(F(x_{n-2})-F(x_{n-2}))+\dots+(F(x_1)-F(x_0))$$

$$=\sum_{i=1}^{n}(F(x_i)-F(x_{i-1})). \tag{3.3.1}$$

By the Mean Value Theorem, for every $i=1,2,\dots,n$, there exists a point c in the interval $[x_i-x_{i-1},]$ such that

$$F'(c)=\frac{F(x_1)-F(x_{i-1})}{x_i-x_{i-1}}. \tag{3.3.2}$$

Since $F'(c) = f'(c)$ and $[x_i - x_{i-1},] = \Delta x_i$, it follows that

$$F(x_1) - F(x_{i-1}) = f(c)\Delta x_i . \qquad (3.3.3)$$

Hence, putting equation (3.3.3) into equation (3.3.1),

$$F(b) - F(a) = \sum_{i=1}^{n} f(c)\Delta x_i .$$

Thus, *F(b)-F(a)* is equal to the value of a Riemann sum using the partition *P*, and so must lie between the upper and lower sums for *P*. That is, we have shown that for any partition *P*,

$$L(f,P) \le F(b) - F(a) \le U(f,P).$$

But, since f is integrable, there is only one number hat has this property, which is $\int_a^b f(x)dx$. Then we have shown that

$$\int_a^b f = F(b) - F(a). \ \square$$

Example 3.3.3: We shall compute the area under the straight line $y= f(x) = x$ between $x = 0$ and $x= 1$. That is $\int_0^1 xdx$.

Solution: An antiderivative of the function *f(x)*= *x* is given by the function $F(x) = \frac{x^2}{2}$. (Can check with the *F'(x)* = *f(x)*.) Now applying the Fundamental Theorem of Calculus, we have

$$\int_0^1 x = F(1) - F(0) = \frac{1}{2}.$$

This is in fact the same results as we found those the laborites process of summing.

Example 3.3.4: Since $F(t) = \frac{2}{3}t^{\frac{3}{2}}$ is an antiderivative of $f(t) = \sqrt{t}$, we have,

$$\int_{0}^{4} \sqrt{t}dt = \frac{2}{3}t^{\frac{3}{2}} |_0^4 = \frac{16}{3} - 0 = \frac{16}{3}.$$

Example 3.3.5: Find the value of $\int_0^2 x^2$.

Solution: We shall proceed numerically,. First, we partition [0, 2] into n equal subdivisions of length $\frac{2}{n}$ each, to obtain the partition

$$P_n = \left\{0, \frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \ldots, \frac{2i}{n}, \ldots, \frac{2n}{n} = 2\right\}.$$

For this partition,

$$U(P_n) = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right)$$

$$= \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

Where the last inequality is obtained by applying the formula for the sum of squares up to n. the sequence $\{U(P_n)\}$ is monotonically decreasing and bounded below. By taking the limit as n tends to infinity, we get

$$\inf\{U(P_n)\} = \lim_{n\to\infty} U(P_n) = \frac{16}{3} = \frac{8}{3}.$$

On the other hand, evaluating $L(P_n)$ leads to

$$L(P_n) = \sum_{i=1}^{n-1} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) = \frac{8}{n^3} \sum_{i=0}^{n-1} i^2 = \frac{8}{n^3} \cdot \frac{n(n-1)(n)(2n-1)}{6}.$$

On taking the limits of $L(P_n)$ as n tends to infinity we get quantity $\frac{8}{3} = \sup L(P_n)$, and now,

$$\frac{8}{3} = \inf\{U(P_n)\} \geq \overline{\int_0^2} x^2 \quad \text{and} \quad \frac{8}{3} = \sup c \leq \underline{\int_0^2} x^2.$$

Hence, $\overline{\int_0^2} x^2 = \frac{8}{3} = \underline{\int_0^2} x^2$, and we conclude that $\int_0^2 x^2 = \frac{8}{3}$.

Discussion: In the calculation above, we used a rather special collection of partitions where each subdivision was of equal length. We would have computed the supremum of $L(f; P)$ and the infimum of $U(f, P)$ for possible partitions, P, of [0, 2], and identified both these quantities with the number $\frac{8}{3}$. Instead, we use the subcollection $\{P_n\}$ consisting of all partition that generated subintervals of equal length, $d = \frac{2}{n}$. For these partitions, we were able to show that

$$\inf\{U(P_n)\} = \sup\{(L(P_n)\} = \frac{8}{3}.$$

With the Definition 4.1.5, is sufficient to guarantee that x^2 is not only integrable, but to establish value of the integral on [0, 2].

While it is the case of f is integrable on $[a, b]$, using equally space for calculation will yield the value of the integral. They exhibit the two important features:

a) The partitions used consist of $(n + 1)$ equally spaced points.

b) The components of the upper and lower sums associated with suprema and imfima over subintervals have been replaced by values obtained by evaluating the function at the endpoints of the subintervals.

In conclusion it is clear that a computation that involves equal length subintervals and evaluation of the function only at the end points of these subintervals.

Discussion: As can be seen from these examples, the Fundamental Theorem of Integral Calculus provides us with powerful tool for evaluating definite integrals exactly. However, to utilize the theorem we must first find an antiderivative for the function we are integrating. But it does not tell us which functions have antiderivatives.

3.4 Improper Integral

All functions that are Riemann integrable have bounded domains and bounded ranges. That functions have bounded function arises from the fact that each of the integral definitions required the function to be defined on a close interval. That function have bounded ranges arose as a prerequisite in the case of a Riemann integrable due to the fact that we require the supremum and infimum of the function to exist as a finite number on each subinterval. In Riemann integration, the integrals are defined for bounded functions over bounded closed interval. The object is to extend the definitions of the Riemann integral $\int_a^b f(x)dx$ to integrals over unbounded intervals for function which are unbounded at a point in the finite interval of definition.

In calculus, an improper integral is the limit of a definite integral as an endpoints of the interval of integration approaches either a specified real number or ∞ or $-\infty$, or in some cases, as both endpoints approach limits. Specifically, an improper integral is a limit of the form

$$\lim_{b\to\infty}\int_a^b f(x)dx \quad , \qquad \lim_{a\to-\infty}\int_a^b f(x)dx ,$$

or of the form

$$\lim_{c\to b^-}\int_a^c f(x)dx \quad , \qquad \lim_{c\to a^+}\int_c^b f(x)dx .$$

In which one takes a limit in one or the other (or sometimes both) endpoints. Improper integrals may also occur at an interior point of the domain of integration, or at multiple such points. It often necessary to use improper integral in order to compute a value for integrals which may not exist in the conventional sense because of a singularity in the function, or an infinite endpoint of the domain of integration.

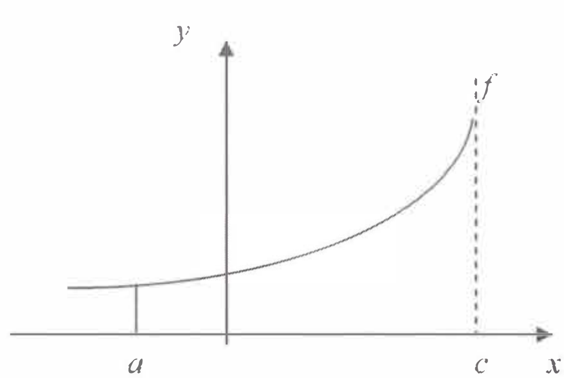


Figure 10: Improper integral

Definition 3.4.1: Let f be a real-valued function on $[a,\infty)$ that is Riemann integrable on $[a, c]$ for every $c > a$. The improper Riemann integral of f on $[a, b)$, denoted by $\int_a^\infty f$, is defined to be

$$\int_a^b f = \lim_{c\to\infty} \int_a^c f$$

provided the limit exists, then the improper integral is said to be convergent. Otherwise, the improper integral is said to be divergent.

Definition 3.4.2: Let f be a real-valued function on $(a, b]$ such that $f \in R[a,b]$ for every $c \in (a,b)$. The improper Riemann integral of f on $(a, b]$, denoted by $\int_a^b f$, is defined to be

$$\int_a^b f = \lim_{c\to a^+} \int_c^b f$$

provided the limit exists, then the improper integral is said to be convergent. Otherwise, the improper is said to be divergent.

Definition 3.4.3 (Convergence of the integral): An improper integral converges if the limit defining it exists. Thus for example one says that the improper integral

$$\lim_{t\to\infty} \int_a^t f(x)dx$$

exists and is equal to L if the integrals under the limit exist for all sufficiently large t, and the value of the limit is equal to L. It is also possible for an improper integral to diverge to infinity. In that case, one may assign the value of ∞ or $-\infty$ to the integral. For instance

$$\lim_{b\to\infty} \int_1^b \frac{1}{x} dx .$$

However, other improper integrals may simply diverge in no particular direction, such as

$$\lim_{b\to\infty} \int_1^b x \sin x dx$$

which does not exist, even as an extended real number. A limitation of the technique of improper integration is that the limit must be taken with respect to one endpoint at a time. Thus, for instance, an improper integral of the form

$$\int_{-\infty}^{\infty} f(x)dx$$

is defined by taking two separate limits, that is

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\lim_{b\to-\infty}\int_{b}^{a} f(x)dx$$

provided the double limit is finite. By the properties of the integral, this can also be written as a pair of distinct improper integrals of the first kind,

$$\lim_{a\to-\infty}\int_{a}^{c} xdx + \lim_{b\to-\infty}\int_{c}^{b} xdx$$

where c is any convenient point at which to start the integration. It is sometimes possible to define improper integrals where both endpoints are infinite.

Definition 3.4.4: If a function f *is* the Riemann integrable for every $s>a$, and we let $F=\int_a^{\infty}|f(x)|dx$, and if F is bounded above on $[a,\infty)$, then $\lim_{s\to\infty} F(s)$ exists and hence $\int_a^{\infty} f(x)dx$ is said to be converge absolutely.

Theorem 3.4.5: The $\int_a^{\infty}\frac{1}{x^p}dx$ for $x>a>0$ converges for $p>1$ and diverges for $p\le 1$.

Proof: The function $f(x)=\frac{1}{x^p}$ is continuous for any $x>a$ and

$$F(s)=\int_a^s \frac{1}{x^p}dx = \frac{1}{1-p}\left(\frac{1}{s^{p-1}}-\frac{1}{a^{p-1}}\right).$$

Hence,

$$\lim_{s\to\infty} F(s) = \frac{-1}{a^{p-1}(1-p)}.$$

If $p>0$ and

$$\lim_{s\to\infty} F(s) = \infty \text{ if } p < 1. \text{ When } p = 1,$$

we get

$$F(s) = \int_a^s \frac{dx}{x} = \log \frac{s}{a}$$

which tends to ∞ as $s \to \infty$. Hence, the given integral converges for $p > 1$ and diverges for $p = 1$ and for $p < 1$. □

Theorem 3.4.6: Let $F(s) = \int_a^\infty f(x)dx$ and $G(s) = \int_a^\infty g(x)dx$ are both convergent, and the $\lim_{s\to\infty} F(s)$ and $\lim_{s\to\infty} G(s)$ exist. Then $\int_a^\infty [f(x) \pm g(x)]dx$ is convergent and

$$\int_a^\infty [f(x) \pm g(x)]dx = A \pm B = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx.$$

Proof: Let

$$\lim_{s\to\infty} F(s) = A \quad \text{and} \quad \lim_{s\to\infty} G(s) = B.$$

Now,

$$\int_a^\infty [f(x) \pm g(x)]dx = \lim_{s\to\infty} \int_a^s [f(x) \pm g(x)]dx$$

$$= \lim_{s\to\infty} \left[\int_a^s [f(x) \pm g(x)]dx \right] = \lim_{s\to\infty} \int_a^s f(x)dx \pm \lim_{s\to\infty} \int_a^s g(x)dx$$

$$= A \pm B.$$

This proved that $\int_a^\infty [f(x) \pm g(x)]dx$ is convergent and

$$\int_a^\infty [f(x) \pm g(x)]dx = A \pm B = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx. \quad \square$$

Theorem 3.4.7 (Comparison Test): If $0 \le f(x) \le g(x)$ for all $x \in [a, \infty)$ and if *f(x)* and *g(x)* are the Riemann integrable on $[a, \infty)$, then

i. If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

ii. If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

Proof: For each *s>0*, we have $0 \le \int_a^s f(x) \le \int_a^s gx)$ which give

$$0 \le F(x) \le G(x).$$

The function *F(s)* and *G(s)* are monotonic increasing function of *s.* hence, if *G(s)* tends to a limit as $s \to \infty$, then *F(s)* will not tend to a limit. Hence if $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.

ii) Prove, if *F(s)* is unbounded so that *G(s)* is also unbounded. Hence if $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ is divergent. □

Example 3.4.8: The function $f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} I$ is locally integrable and derivative of

$$F(x) = x^2 \sin\frac{1}{x}$$

on $\left[\frac{-2}{\pi}, 0\right)$. Hence,

$$\int_{\frac{-2}{\pi}}^{s} f(x)dx = x^2 \sin\frac{1}{x}\Big|_{\frac{-2}{\pi}}^{s} = s^2 \sin\frac{1}{s} + \frac{4}{\pi^2}$$

and

$$\int_{\frac{-2}{\pi}}^{s} f(x)dx = \lim_{s \to 0^-}\left(s^2 \sin\frac{1}{s} + \frac{4}{\pi^2}\right) = \frac{4}{\pi^2}$$

according to Definition 3.4.1. However, this is not an improper integral, even though *f(0)* is not defined and cannot be defined so as to make *f* continuous at 0. If we define *f(0)* arbitrarily, then f is bounded on the closed interval $\left[\frac{-2}{\pi},0\right)$ and continuous except at 0. Therefore $\int_{\frac{-2}{\pi}}^{s} f(x)dx$ exists and equals $\frac{4}{\pi^2}$ as a proper integral.

Example 3.4.9: We have some example for the first kind of the first kind of the Definition 3.4.1 are

i. $\int_1^\infty \frac{1}{x^2}dx$

ii. $\int_1^\infty \frac{1}{\sqrt{x}}dx$

Solution:

i $\int_1^\infty \frac{1}{x^2}dx$ is convergent, then $F(s)=\int_1^s \frac{1}{x^2}dx = 1-\frac{1}{s}$ so that

$$\lim_{s\to\infty}\left(1-\frac{1}{s}\right)=1$$

Hence, $\int_1^\infty \frac{1}{x^2}dx$ is convergent and $\int_1^\infty \frac{1}{x^2}dx = 1$

ii $\int_1^\infty \frac{1}{\sqrt{x}}dx$ is divergent. Now,

$$F(s)=\int_1^{\infty,s} \frac{1}{\sqrt{x}}dx = 2(\sqrt{s}-1)\backslash$$

So, $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} 2(\sqrt{s}-1)$ is infinite. Hence, $\int_1^\infty \frac{1}{\sqrt{x}}dx$ is divergent.

Example 3.4.10: (Theorem 4.4.3)

i. $\int_a^\infty \frac{1}{e^x+1} dx$

Solution: For every x in $[0,\infty)$, we have $\frac{1}{e^x+1} < \frac{1}{e^x}$;

Let $f(x) = \frac{1}{e^x+1}$ and $g(x) = \frac{1}{e^x}$. While, we obtained that $\int_0^\infty e^{-x} dx$ is convergent and $\int_0^\infty e^{-x} dx = 1$. Hence, by comparison test, $\int_a^\infty g(x) dx$ is convergent, then $\int_a^\infty f(x) dx$ is also convergent.

3.5 Riemann-Stieltjes Integral

The Riemann-Stieltjes integral is a generalization of Riemann integral. It is defined that the Riemann-Stieltjes integral of a real-valued function *f* of a real variable with respect to a real function *g* is denoted by

$$\int_a^b f(x)g(x),$$

and defined to be the limit, as the mesh of the partition *P* of the interval $[a, b]$ approaches zero, of the approximating sum

$$\sum_{x_i} f(c_i)(g(x_{i+1}) - g(x_i)).$$

Where c_i is in the *i*-th subinterval $[x_i, x_{i+1}]$. The two function *f* and *g* are respectively called the integrand and the integrator. Most commonly, *g* will be non-decreasing, but this is not required. In order that is Riemann-Stieltjes integral exists it is necessary that *f* and *g* do not share any points of discontinuity. An alternative, and slightly more general, definition of the Riemann-Stieltjes integral uses the same approximating sums. And it takes the limits as more and more division points are inserted into the partition of $[a, b]$. With this definition, an integral can exist when *f* and *g* share points of discontinuity, as long as they are not discontinuous from the same side at the same point.

If g is differentiable to everywhere, then the integral may still be different from the Riemann integral.

$$\int_a^b f(x)g'(x)dx.$$

They will be same if the derivative is unbounded and continuous. However, when the functions are continuous and increasing, g, such as Cantor function may have jump discontinuity and have derivative zero almost everywhere. For the form of integration by parts of the Riemann-Stieltjes integral, the existence of the integral on the left implies the existence of the integral on the right.

$$\int_a^b f(x)dg(x) = f(b) - g(b) - f(a)g(a) - \int_a^b g(x)df(x).$$

Definition 3.5.1 (Definition of the Riemann-Stieltjes Integral): Let α be a monotone increasing function on $[a, b]$, and let f be a bounded real-valued function on $[a, b]$. For each partition $P = \{x_0, x_1, \ldots\ldots, x_n\}$ of $[a, b]$, set

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \qquad i = 1, 2, \ldots, n.$$

Since g is monotone increasing, $\Delta\alpha \geq 0$ for all i, let

$$m_i = \inf\{f(t) : x_{i-1} \leq t \leq x_i\} \qquad \text{and} \qquad M_i = \sup\{f(t) : x_{i-1} \leq t \leq x_i\}.$$

As for the Riemann integral, the upper Riemann-Stieltjes sum of f over the partition P with respect to α is, denoted $U(f, P, \alpha)$, is defined by

$$U(f, P, \alpha) = \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})).$$

Similarly, the lower Riemann-Stieltjes sum of f with respect to α and the partition P, denoted $L(f, P, \alpha)$, is defined by

$$L(f, P, \alpha) = \sum_{i=1}^{n} m_i(\alpha(x_i) - \alpha(x_{i-1})).$$

Since $m_i \leq M_i$ and $\Delta\alpha_i \geq 0$, we have $L(f, P, \alpha) \leq U(f, P, \alpha)$ for any bounded function f and any partition P. Furthermore, if $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m[\alpha(b) - \alpha(a)] \leq L(f, P, \alpha) \leq U(f, P, \alpha) \leq M[\alpha(b) - \alpha(a)]$$

for all partition P of $[a, b]$. Let P be any partition of $[a, b]$. Since $M_i \leq M$ for all i and $\Delta\alpha_i \geq 0$,

$$\sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} M\Delta\alpha_i = M\sum_{i=1}^{n} \Delta\alpha_i = M[\alpha(b)-\alpha(a)].$$

Thus, $U(f,P,\alpha) \le M[\alpha(b)-\alpha(a)]$. The other inequality follows similarly. In the above we have used the fact that

$$\begin{aligned}\sum_{i=1}^{n} \Delta\alpha_i &= (\alpha(x_1)-\alpha(x_0)) + (\alpha(x_2)-\alpha(x_1)) + \dots\dots + (\alpha(x_n)-\alpha(x_{n-1})) \\ &= \alpha(x_n)-\alpha(x_0) = \alpha(b)-\alpha(a).\end{aligned}$$

In analogy with the Riemann integral, the upper and lower Riemann-Stieltjes integrals of f with respect to α over $[a, b]$, denoted $\overline{\int_a^b} f d\alpha$ *and* $\underline{\int_a^b} f d\alpha$ respectively, are defined by

$$\overline{\int_a^b} f d\alpha = \inf\{U(f,P,\alpha) : P \text{ is a partition } [a,b]\},$$

$$\underline{\int_a^b} f d\alpha = \sup\{L(f,P,\alpha) : P \text{ is a partition } [a,b]\}.$$

The $\{U(f,P,\alpha) : P$ is a partition $[a,b]\}$ is bounded below, and thus the upper integral of f with respect to α exists as a real number. Similarly, the lower sums are bounded above, and thus the supremum defining the lower integral is also finite.

Definition 3.5.2: Let f be a bounded real-valued function on $[a, b]$, and α a monotone increasing function on $[a, b]$. If

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$$

then f is said to be Riemann-Stieltjes integrable or intrgrable with respect to α on $[a, b]$. The common value is denoted by

$$\int_a^b f d\alpha \quad or \quad \underline{\int_a^b} f(x) d\alpha(x)$$

and is called as Riemann-Stieltjes integral of f with respect to α.

Theorem 3.5.3: Let f be a bounded real-valued function on $[a, b]$, and α a monotone increasing function on $[a, b]$. Then

$$\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha .$$

Proof: With the proof of Definition 3.1.5, if P^* is a refinement of the partition P, then

$$L(f,P,\alpha) \le L(f,P^*,\alpha) \le U(f,P^*\alpha) \le U(f,P,\alpha).$$

Thus if P, L are any two partition of $[a, b]$,

$$L(f,P,\alpha) \le L(f,P\cup L,\alpha) \le U(f,P\cup L,\alpha) \le U(f,P,\alpha).$$

Therefore $L(f,P,\alpha) \le U(f,P,\alpha)$ for any two partition P, L. Hence,

$$\underline{\int_a^b} f d\alpha = \sup_P L(f,P,\alpha) \le U(f,P,\alpha)$$

for any partition L. □

Theorem 3.5.4: Let α be a monotone increasing on $[a, b]$. A bounded real-valued function f is Riemann-stieltjes integrable with respect to α on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon .$$

Furthermore, if P is a partition of $[a, b]$ for which the above holds, then the inequality also holds for all refinement of P.

Proof:

The proven are similarly with the theorem 3.1.7. □

Theorem 3.5.5: Let f be a real-valued function on $[a, b]$ and α a monotone increasing function on $[a, b]$.

i) If f is continuous on $[a, b]$, then f is integrable with respect to α on $[a, b]$.

ii) If f is monotone on $[a, b]$, and α is continuous on $[a, b]$, then f is integrable with respect to α on $[a, b]$.

Proof: i) Let $\varepsilon > 0$, choose $\dfrac{\varepsilon}{\alpha(b)-\alpha(a)} > 0$ such that

$$[\alpha(b)-\alpha(a)]\frac{\varepsilon}{\alpha(b)-\alpha(a)} < \varepsilon .$$

Since *f* is continuous on [*a*, *b*], thus exists $\delta > 0$ such that

$$| f(x) - f(y) | < \frac{\varepsilon}{\alpha(b)-\alpha(a)} \tag{3.5.1}$$

for all $x, y \in [a,b]$ with $(x-y) < \delta$. Choose a partition P of [*a*, *b*] such that $\Delta\alpha_i < \delta$ for all $i = 1,2,\ldots,n$. Then by inequality (3.5.1),

$$M_i\Delta\alpha_i - m_i\Delta\alpha_i \le \frac{\varepsilon}{\alpha(b)-\alpha(a)}$$

for all $i = 1,2,\ldots,n$.

Therefore,

$$U(f,P,\alpha) - L(f,P,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i$$

$$\le \frac{\varepsilon}{\alpha(b)-\alpha(a)}\sum_{i=1}^{n}\Delta\alpha_i = \frac{\varepsilon}{\alpha(b)-\alpha(a)}(\alpha(b)-\alpha(a)) < \varepsilon .$$

Thus then f is intgerable with respect to α on [*a*, *b*].

for any positive integer *n*, choose a partition $P = \{x_0, x_1, \ldots\ldots, x_n\}$ of [*a*, *b*] such that

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{1}{n}[\alpha(b)-\alpha(a)].$$

Since g is continuous, assume *f* is monotone increasing on [*a*, *b*]. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore,

$$U(f,P,\alpha) - L(f,P,\alpha) = \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]\Delta\alpha_i$$

$$= \frac{[\alpha(b)-\alpha(a)]}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$$

$$= \frac{[\alpha(b)-\alpha(a)]}{n}\cancel{[\alpha(b)-\alpha(a)]}.$$

Given $\varepsilon > 0$, choose $n \in \mathrm{N}$ such that

$$\frac{[\alpha(b)-\alpha(a)]}{n}[f(b) - f(a)] < \varepsilon .$$

For this n and corresponding partition P, $U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon$, which proves the results. □

Definition 3.5.6: For a given monotone increasing function α on $[a, b]$, $R(\alpha)$ denoted the set of bounded real-valued function f on $[a, b]$ that are Riemann-Stieltjes integrable with respect to α.

Theorem 3.5.7 (Mean Value Theorem): Let f be a continuous real-valued function on $[a, b]$. Then there exists $c \in [a,b]$ such that

$$\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha(a)].$$

Proof: Let m and M denote the minimum and maximum of f on $[a, b]$ respectively. Then by Theorem 3.6.4(d),

$$m[\alpha(b) - \alpha(a)] \le \int_a^b f d\alpha \le M[\alpha(b) - \alpha(a)].$$

If $\alpha(b) - \alpha(a) = 0$, then any $c \in [a,b]$ will work. If $\alpha(b) - \alpha(a) \ne 0$, then by the intermediate value theorem there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{\alpha(b) - \alpha(a)} \int_a^b f d\alpha. \ \square$$

Theorem 3.5.8 (Integration by Parts): Suppose α and β are monotone increasing function ob $[a, b]$ and that each is Stieltjes integrable with respect to the other. Then

$$\int_a^b \beta d\alpha = \beta(b)\alpha(b) - \beta(a)\alpha(a) - \int_a^b \alpha d\beta.$$

Proof: Since α and β are both increasing, for any partition $P = \{x_0, x_1, \ldots, x_n\}$ on $[a, b]$, we have

$$U(\beta, P, \alpha) + L(\alpha, P, \beta) = \sum_{i=1}^{n} \beta(x_i)(\alpha(x_i) - \alpha(x_{i-1})) + \sum_{i=1}^{n} \alpha(x_i)(\beta(x_i) - \beta(x_{i-1}))$$

$$= \sum_{i=1}^{n} (\beta(x_i)(\alpha(x_i) - \beta(x_i)\alpha(x_{i-1}) + \alpha(x_{i-1})\beta(x_i) - \alpha(x_{i-1})\beta(x_{i-1})$$

$$= \sum_{i=1}^{n} (\beta(x_i)\alpha(x_i) - \alpha(x_{i-1})\beta(x_i)) = \beta(b)\alpha(b) - \beta(a)\alpha(a). \qquad (3.5.2)$$

From the integrability of α and β with respect to each other for every $\varepsilon > 0$, there exists a partition P such that

$$\left| L(\alpha, P, \beta) - \int_a^b \alpha d\beta \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| U(\beta, P, \alpha) - \int_a^b \beta d\alpha \right| < \frac{\varepsilon}{2},$$

which together with equation (3.5.2) implies that,

$$\left| \int_a^b \alpha d\beta + \int_a^b \beta d\alpha - \beta(b)\alpha(b) + \beta(a)\alpha(a) \right| < \varepsilon .$$

Since this holds for every $\varepsilon > 0$, the left side must be 0. □

Example 3.5.9: Discuss the Riemann-Stieltjes Integral integrability of the function, *f(x)=x*, $x \in [0,2]$, with respect to the function , $\alpha : [0,2] \to R$ defined by

$$\alpha(x) = \begin{cases} 0, & if\ x \in [0,1], \\ 1, & if\ x \in (1,2]. \end{cases}$$

Solution : Given $\varepsilon > 0$, choose a partition, P, of $[0, 2]$ such that $\| P \| < \varepsilon$. Notice that if $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \neq 0$, then $[1, \delta) \subseteq [x_i, x_{i-1}]$, for some $\delta > 1$. It follows that there is exactly one i such that $\Delta\alpha_i \neq 0$. For this i, we have

$$\begin{aligned} |U(f, P, \alpha) - L(f, P, \alpha)| &= \left| \sum_{i=1}^n M_i \Delta\alpha_i - \sum_{i=1}^n m_i \Delta\alpha_i \right| \\ &= M_i \Delta\alpha_i - m_i \Delta\alpha_i \\ &= (M_i - m_i)(1-0) = x_{i+1} - x_i \\ &\leq \| P \| < \varepsilon . \end{aligned}$$

Since ε was arbitrary, the existence of the integrals allows, and its value is easily seen to be 1.

Discussion: This example illustrates one of the useful properties of Stieltjes integrals, with their difference 'weights' to different values of *f*. In this case, the only points of *f* that is important for purposes of computing this integral is the value at 1. This is due to the fact that α is constant on any interval which does not include 1, while at 1, α undergoes rapid change.

Example 3.5.10: Discuss the Riemann-Stieltjes integral integrabiliy of the function $f:[0,2]\to R$ defined by

$$\alpha(x)=\begin{cases}0, & \text{if } x\in[0,1],\\ 1, & \text{if } x\in(1,2].\end{cases}$$

With respect to the function $\alpha(x)=f(x)$.

Solution: For any partition P^*, there exists a refinement $P=\{0=x_0\le x_1\le .....\le x_n=2\}$ of $[0,2]$, for which there exists a subinterval $[x_i,x_{i-1}]$ that contains the point 1 as well as a point greater than 1. As in the example above, we have $\Delta\alpha_i=1$ while $\Delta\alpha_j=0$ for $i\neq j$, whence

$$U(f,P,\alpha)=\sum_{i=1}^{n}M_i\Delta\alpha_i=M_i\Delta\alpha_i=1(1-0)=1.$$

On the other hand,

$$L(f,P,\alpha)=\sum_{i=1}^{n}m_i\Delta\alpha_i=m_i\Delta\alpha_i=0(0-1)=0.$$

Consequently, $\overline{\int_0^1} fd\alpha=1$ and $\underline{\int_0^1} fd\alpha=0$. Hence, $\int_0^1 fd\alpha$ does not exist.

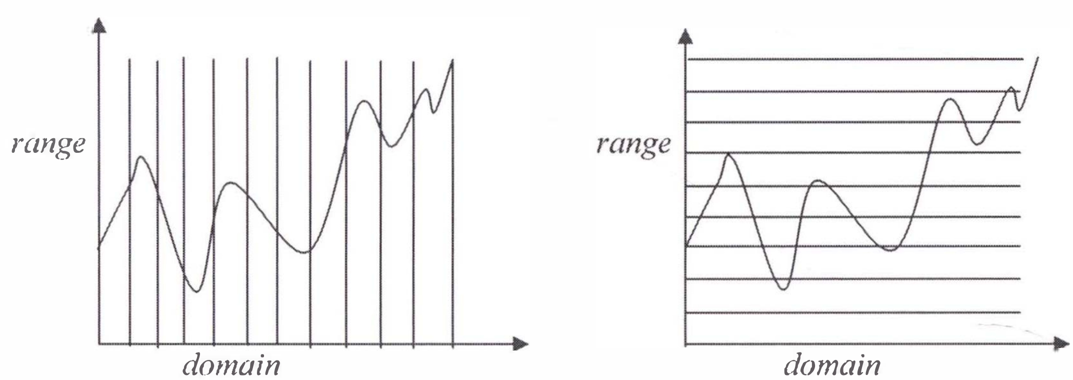
Discussion: α does not have to be continuous. $\Delta\alpha_i$ does not have to shrink to 0 as $\|P\|\to 0$. Now Theorem 4.6.3(a) uses the fact that $|f(x_i)-f(x_{i-1})|$ must shrink to 0 as the norm of P goes to 0 to avoid the problems with discontinuities in α. Theorem 4.6.3(b) uses the continuity of α, which forces $\Delta\alpha_i$ to shrink to 0 as the norm of P goes to 0 to avoid the difficulties with discontinuous and the points of discontinuity match in such a way that the situation cannot be retrieved.

CHAPTER 4

THE LEBESGUE INTEGRAL

4.1 Measure Theory

Rather than partitioning the domain of the function, as in the Riemann integral, Lebesgue choose to partition the range. Thus, for each interval partition, rather than asking the value of the function between the end points of the interval in the domain, the Lebesgue asked how much of the domain is mapped by the function to some value between two end points in the range.



Figurel 1: Two ways to counts rectangles- partitioning the range as opposed to partitioning the domain of a function.

Measure theory was used to determining how much of the domain is sent to a particular portion of a partition of the range. The notion of measure is based on capturing the essence of a simple intuitive idea and extending it by a mathematical procedure to more general setting. The intuitive idea in our case is the length of m, denoted by $m(A)$, which is the difference between its end-points.

Definition 4.1.1: If J is an interval, we define the measure of J, denoted $m(J)$, to be the length of J. Thus if J is (a, b), $(a, b]$, $[a, b)$, or $[a, b]$, $a, b \in R$, then

$$m(J) = b - a.$$

Definition 4.1.2: If A is an open subset of R, then there exists a finite or countable collection $\{I_n\}$ of pairwise disjoint open intervals such that

$$A = \bigcup_n I_n.$$

Recall, the family $\{I_n\}$ is pairwise disjoint if and only if $I_n \cap I_m = \emptyset$ whenever $n \neq m$.

Definition 4.1.3: If A is an open subset of R with $A = \bigcup_n I_n$ where $\{I_n\}$ is a finite or countable collection of pairwise disjoint open intervals, we define the measure of A, denoted $m(A)$, by

$$m(A) = \sum_n m(I_n).$$

Remark:

(a) For the empty set $\emptyset$, we set $m(A)=0$.

(b) The sum defining $m(A)$ may be either finite of infinite. If any of the intervals are of infinite length, then $m(A) = \infty$. On the other hand, if

$$A = \bigcup_{n=1}^{\infty} I_n,$$

where the I_n are pairwise disjoint bounded open intervals, we may still have

$$m(A) = \sum_{n=1}^{\infty} m(I_n) = \infty,$$

due to the divergence of the series to ∞. Since $m(I_n) \geq 0$ for all n, the sequence of partial sums is monotone increasing and thus either convergence to a real number or divergence to ∞.

Theorem 4.1.4: If A and B are open subsets of R with $A \subset B$, then

$$m(A) \leq m(B).$$

Proof: Suppose $A = \bigcup_n I_n$ and $B = \bigcup_n J_m$ where $\{I_n\}_n$ and $\{J_m\}_m$ are finite or countable collection of pairwise disjoints open intervals. Since $A \subset B$, each interval $I_n \subset J_m$ for some m, let

$$N_m = \{n : I_n \subset J_m\}.$$

Since the collection $\{J_m\}_m$ is pairwise disjoint, so is the collection $\{N_m\}_m$ and

$$A = \bigcup_n I_n = \bigcup_m \bigcup_{n \in N_m} I_n.$$

Therefore,

$$m(A) = \sum_m \sum_{n \in N_m} m(I_n).$$

Then,

$$\sum_{n \in N_m} m(I_n) \leq m(J_m). \quad \square$$

Remark: If A is an open subset of (a, b), $a, b \in R$, then $. m(A) \leq b - a$. Thus every bounded open set has finite measure.

Theorem 4.1.5: If A is an open subset of R, then

$$m(A) = \lim_{n \to \infty} m(A_n)$$

where for each $n \in \mathrm{N}, A_n = A \cap (-n, n)$.

Proof: For each n, A_n is open, with

$$A_n \subset A_{n+1} \subset A$$

for all $n \in \mathrm{N}$. By Theorem 4.1.4, the sequence $\{m(A_n)\}$ is monotone increasing with $m(A_n) \leq m(A)$ for all n. Therefore,

$$\lim_{n \to \infty} m(A_n) \leq m(A). \tag{4.1.1}$$

If A is bounded, then there exists $n_0 \in \mathrm{N}$ such that

$$A \bigcap (-n, n) = A$$

for all $n \geq n_0$. Hence $m(A_n) \leq m(A)$ for all $n \geq n_0$ and thus equality holds in equation (4.1.1). Suppose that A is an unbounded open subset of R with

$$A = \bigcup_{n=1} I_n$$

where $\{I_n\}$ is a finite or countable collection of pairwise disjoint open intervals. If $m(I_n) = \infty$ for some n, either $I_n = R$ or I_n is an interval of the form $(-\infty, a_n)$ or (a_n, ∞) for some $a_n \in R$. Suppose $I_n = (a_n, \infty)$. Choose $n_0 \in \mathrm{N}$ such that $n_0 \geq |a_n|$. Then for all $n \geq n_0$,

$$I_n \bigcap (-n, n) = (a_n, n),$$

and thus,

$$\infty = \lim_{n \to \infty} m(I_n \bigcap (-n, n)) \leq \lim_{n \to \infty} m(A_n) \leq m(A).$$

Therefore holds the equation (4.1.1). Suppose $m(I_n) < \infty$ for all n. Since A is unbounded, the collection $\{I_n\}$ must be infinite. If the collection were finite, then since each interval has finite length, each intervals is bounded, and as a consequence A must also be bounded. Let $\alpha \in R$ with $\alpha < m(A)$. Since

$$\sum_{n=1}^{\infty} m(I_n) = m(A) > \alpha,$$

there exists a positive integral N such that

$$\sum_{n=1}^{N} m(I_n) > \alpha.$$

Let $B = \bigcup_{n=1}^{N} I_n$. Then B is a bounded open set, and thus by the above,

$$m(B) = \lim_{n \to \infty} m(B \bigcap (-n, n)).$$

Since $m(B) > \alpha$, there exists $n \geq n_0$ such that

$$m(B \bigcap (-n, n)) > \alpha \quad \text{for all } n \geq n_0.$$

But $B \bigcap (-n, n) \subset A_n$ for all $n \in N$. Hence by Theorem 4.1.4,

$$m(B \bigcap (-n, n)) \leq m(A_n),$$

and as a consequence

$$m(A_n) > \alpha \quad \text{for all } n \geq n_0.$$

If $m(A)=\infty$, then since $\alpha < m(A)$ was arbitrary, we have $m(A_n) \to \infty$ as $n \to \infty$. If $m(A) < \infty$, then give $\varepsilon > 0$, take $\alpha = m(A) - \varepsilon$. By the above, there exists $n_0 \in N$ such that

$$m(A) - \varepsilon < m(A_n) \leq m(A) \text{ for all } n \geq n_0.$$

Therefore,

$$m(A) = \lim_{n\to\infty} m(A_n). \ \square$$

Theorem 4.1.6: If $\{A_n\}_n$ is a finite or countable collection of open subsets of R, then

$$m\left(\bigcup_n A_n\right) \leq \sum_n m(A_n).$$

Proof: If $\{I_n\}_{n=1}^N$ is a finite collection of bounded open intervals, then

$$m\left(\bigcup_{n=1}^N I_n\right) \leq \sum_{n=1}^N m(I_n).$$

The collection $\{I_n\}$ is not assumed to be pairwise disjoint. □

Definition 4.1.7: If E is a subset of R, the characteristic function of E, denoted χ_E, is the function defined by

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Suppose I is a bounded and open interval. Choose $a, b \in R$ such that $I \subset [a,b]$. Since χ_I is continuous on $[a, b]$ except at the two endpoints of I, $\chi_I \in R[a,b]$ with

$$\int_a^b \chi_I(x)dx = m(I).$$

If A is an open subset of $[a, b]$ with

$$A = \bigcup_{n=1}^m J_n$$

where $\{J_n\}$ are pairwise disjoint open intervals, then

$$\chi_E(x) = \sum_{n=1}^m \chi_{J_n}(x),$$

and thus,

$$m(A) = \sum_{n=1}^{m} m(J_n) = \sum_{n=1}^{m} \int_a^b \chi_{J_n}(x)dx = \int_a^b \chi_U(x)dx .$$

To defined the measure of a compact of subset of R, let K is a subset compact of R and A is any bounded and open set containing K, then

$$A = K \bigcup (A \setminus K).$$

The fact of $A \setminus K$ is also open and bounded, and thus has finite measure, the measure of K will be defined.

Definition 4.1.8: Let K be a compact subset of R. the measure of K, denoted $m(K)$, is defined by

$$m(K) = m(A) - m(A \setminus K),$$

where A is any bounded and open subset of R containing K.

Theorem 4.1.9: If K is compact, then $m(K)$ is well defined.

Proof: Suppose A and B are any two bounded open sets containing K, then we have,

$$\begin{aligned} m(A) + m(B \setminus K) &= m(A \bigcup (B \setminus K)) + m(A \bigcap (B \setminus K)) \\ &= m(A \bigcup B) + m((A \bigcap B) \setminus K). \end{aligned}$$

In the above, we have use the fact that

$$A \bigcup (B \setminus K) = A \bigcup B$$

and

$$A \bigcap (B \setminus K) = (A \bigcap B) \setminus K .$$

Similarly,

$$m(A \setminus K) + m(B) = m(A \bigcup B) + m((A \bigcap B) \setminus K).$$

Therefore,

$$m(A) + m(B \setminus K) = m(B) + m(A \setminus K).$$

Since all the terms are finite,

$$m(A) - m(A \setminus K) = m(B) - m(B \setminus K).$$

Thus the notation of $m(K)$ is independent of the choice of A; $m(K)$ is well-defined. □

Theorem 4.1.10: If A is an open subset of R and $a,b \in R$, then

$$m(A \cap [a,b]) + m(A' \cap [a,b]) = b - a.$$

Proof: When $A' = R \setminus A = \{x \in R : x \notin A\}$. If A is open, the A' is closed and thus $A' \cap [a,b]$ is a compact subset of $[a, b]$. Suppose $B \supset [a,b]$ is open. Let $K = A' \cap [a,b]$. Then since $B \supset K$,

$$m(K) = m(B) - m(B \setminus K).$$

But,

$$B \setminus K = B \cap (A' \cap [a,b])' = (B \cap A) \cup (B \cap [a,b]') \supset B \cap A.$$

Therefore, $m(B \cap A) \le m(B \setminus K)$. Since $A \cap [a,b] \subset A \cap B$,

$$m(B \cap [a,b]) + m(K) \le m(A \cap B) + m(B) - m(B \setminus K) \le m(B).$$

Given $\varepsilon > 0$, take $B = (a - \varepsilon, b + \varepsilon)$. Then

$$m(A \cap [a,b]) + m(A' \cap [a,b])) \le b - a + 2\varepsilon.$$

Since $\varepsilon > 0$, is arbitrary, this proves that

$$m(A \cap [a,b]) + m(A' \cap [a,b])) \le b - a.$$

To prove the reverse inequality, let $I_\varepsilon = [a + \varepsilon, b - \varepsilon]$, where $0 < \varepsilon < \frac{1}{2}(b-a)$. Then

$$m(A \cap [a,b]) + m(A' \cap [a,b])) \ge m(A \cap (a,b)) + m(A' \cap I_\varepsilon).$$

Since (a, b) is an open set containing $A' \cap I_\varepsilon$,

$$m(A' \cap I_\varepsilon) = b - a - m((a,b) \setminus (A' \cap I_\varepsilon)).$$

But,

$$m((a,b) \setminus (A' \cap I_\varepsilon)) = m(((a,b) \cap A) \cup ((a,b) \cap I'_\varepsilon))$$

$$= m(((a,b) \cap A) \cup (a, a - \varepsilon) + (b - \varepsilon, b)).$$

By Theorem 4.1.6,

$$\le m(A \cap (a,b)) + 2\varepsilon.$$

Therefore,

$$m(A \cap [a,b]) + m(A' \cap [a,b]) \ge b - a - 2\varepsilon. \square$$

Definition 4.1.11 (Inner and Outer Measure): Let E be a subset of R. The Lebesgue outer measure of E, denoted $m^*(E)$, is defined by

$$m^*(E) = \inf\{m(A) : A \text{ is open with } E \subset A\}.$$

The Lebesgue inner measure of E, denoted $m^*(E)$, is defined by

$$m_*(E) = \sup\{m(K) : K \text{ is compact with } K \subset E\}.$$

$$m_*(E) = m^*(E) - m^*(E \setminus K).$$

Theorem 4.1.12: The measure m^* and m_* both exhibit monotonicity. That is, given $A \subset B \subset R$, and for any subsets of R, $0 \le m_*(E) \le m^*(E)$ it follows that

$$m^*(E_1) \le m^*(E_2) \quad \text{and} \quad m_*(E_1) \le m_*(E_2).$$

Proof: If K is compact and A is open with $K \subset E \subset A$, then

$$0 \le m(K) \le m(A).$$

If K is fixed, then $m(K) \le m(A)$ for all open sets A containing E. Taking the infimum over all such A gives

$$0 \le m(K) \le m^*(E). \ \square$$

Example 4.1.13 (Definition 4.1.3):

For all $n=1,2,\ldots$,set $I_n = \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right)$. Then

$$I_1 = \left(1 - \frac{1}{2}, 1 + \frac{1}{2}\right), I_2 = \left(2 - \frac{1}{4}, 2 + \frac{1}{4}\right), \ etc.$$

Since $n + 2^{-n} < (n+1) - 2^{-(n+1)}$ for all $n \in N$, the collection $\{I_n\}_{n=1}^{\infty}$ is pairwise disjoint. Let $A = \bigcup_{n=1}^{\infty} I_n$.

Then,

$$m(A) = \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} 2\frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2} = \frac{1}{1-(1/2)} = 2.$$

The set A is an example of unbounded open set with finite measure.

Example 4.1.14 (Theorem 4.1.12)

a) If E is any contable subset of R, then $m_*(E) = m^*(E) = 0$. Suppose $E = \{x_n\}_{n=1}^{\infty}$.

Let $\varepsilon > 0$ be arbitrary. For each n, let $I_n = \left(x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n}\right)$, and set $A = \bigcup_{n=1}^{\infty} I_n$.

Then A is open with $E \subset A$. by Theorem 4.1.6,

$$m(A) \leq \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} = 2\varepsilon .$$

Therefore, $m^*(E) < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $m^*(E) = 0$. As a consequence, we also have $m^*(E) = 0$.

b) If I is any bounded interval, then $m_*(I) = m^*(I) = m(I)$, suppose I=(*a*, *b*) with $a, b \in R$. Since I is open, $m^*(I) \leq m(I) = b - a$. On the other hand, if $0 < \varepsilon < b - a$, then $\left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right]$ is a compact subset of *I*, and as a consequence,

$$b - a - \varepsilon = m\left[a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}\right] \leq m_*(I).$$

Therefore, $b - a - \varepsilon \leq m_*(I) \leq m^*(I) \leq b - a$. Since $\varepsilon > 0$ was arbitrary, equality holds. A similar argument proves that if I is any closed and bounded interval, then $m_*(I) = m^*(I) = m(I)$. As a consequence of Theorem 4.1.12, the result holds for any bounded intervals *I*.

4.2 Measurable Sets and Measurable Functions

Definition 4.2.1: A set $E \subset R$ is Lebesgue measurable, if $m^*(E) = m_*(E)$, measure *E* is denoted simply by m(*E*) and is given by

$$m(\mathrm{E}) = m^*(E) = m_*(E).$$

A straight forward extension of this definition was applies to unbounded sets.

Definition 4.2.2: The measure for an unbounded set E is defined as

$$m(E) = \lim_{n \to \infty} m(E \cap [-n, n]).$$

Remark: If E is unbounded and $E \cap I$ is measurable for every closed and bounded interval *I*, then the sequence $\{ m(E \cap [-n, n]) \}_{n=1}^{\infty}$ is non- decreasing, and as a consequence $m(E) = \lim_{n \to \infty} m(E \cap [-n, n])$ exists.

Theorem 4.2.3: The outer measure, m^*, is countably additive on the set of all measurable subset of R. If $\{A_n \mid n = 1,2,...\}$ is a set of measurable subsets of R, then

$$m^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m_*(A_n).$$

Theorem 4.2.4: Every set E of outer measure zero is measurable with $m(E) = 0$

Proof: Suppose $E \subset R$ with $m^*(E) = 0$. Then for any closed and bounded interval I,

$$m^*(E \bigcap I) \leq I^*(E) = 0.$$

Thus $m_*(E \bigcap I) = m^*(E \bigcap I) = 0$ and hence $E \bigcap I$ is measurable for every closed and bounded interval I. Since $m(E \bigcap [-n,n]) = 0$ for every $n \in N$, $m(E) = 0$. □

Definition 4.2.5 (Measurable function): Let E be a bounded measurable subset of R and $f : E \to R$ a function. Then f is said to be measurable on E if $\{x \in E \mid f(x) > r\}$ is measurable for every real number r. Since $f^{-1}((r,\infty)) = \{x : f(x) > r\}$, f is measurable if and only if $f^{-1}((r,\infty))$ is a measurable set for every $r \in R$.

Theorem 4.2.6: If each function in sequence $\{f_n\}$ is measurable on a set A and if f is the pointwise limit function of $\{f_n\}$, then f is measurable very well.

Proof: Let $x \in E$ and $r \in R$ such that $f(x) > r$. Let p be a natural number such that $f(x) > r + \frac{1}{p}$. Then, by definition of limit, there exists a natural number N such that for all $n > N$,

$$f_n(x) > r + \frac{1}{p}.$$

Thus,

$$f(x) = \lim_{n \to \infty} f_n(x) > r + \frac{1}{p} > r.$$

This implies that

$$\{x \in E \mid f(x) > r\} = \bigcup_{p=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcup_{n=N+1}^{\infty} \{x \in E \mid f_n(x) > r + \frac{1}{p}\}$$

Since this set is measurable and r was arbitrary, then f is measurable. □

Definition 4.2.7: The simple function is the development of the Lebesgue integral and will make use of a pedestrian class of measurable class. A simple function $f : A \to R$ is a measurable function which takes on finitely many values.

Theorem 4.2.8: A function $f : A \to R$ is measurable if and only if it is the pointwise limit of a sequence of a simple function.

Proof: Suppose r is a simple function on $[a, b]$ with range $r = \{\alpha_1, ..., \alpha_n\}$, where $\alpha_1 \neq \alpha_j$ wherever $i \neq j$. For each i=1, … ,n, set

$$A_i = \{x \in [a,b] : r(x) = \alpha_i\} = r^{-1}(\{\alpha_i\}).$$

Since r is measurable, each A_i is a measurable set, and

$$r(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x).$$

Furthermore, since $\alpha_1 \neq \alpha_j$, if $i \neq j$, the set A_i, i=1,…,n, are pairwise disjoint with $\bigcup_{i=1}^{n} A_i = [a,b]$, $r(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x)$ is called the canonical representation of r. If all the set A_i are intervals, then r is a step function on $[a, b]$. □

4.3 Integrating Bounded Measurable Function

To construct the integrating bounded measurable functions, we will first constructing the Riemann integral except partitioning the range rather than the domain of the function.

Let $f : E \to R$ be a bounded measurable function on measurable subset E of R. Let $a = \inf\{f(x) \mid x \in E\}$ and $b > \sup\{f(x) \mid x \in E\}$, u is arbitrary insofar as it is greater than the least upper bound of f on E. We will define the Lebesgue integral of f over an interval E as the limit of Lebesgue sums.

Definition 4.3.1: The Lebesgue sum of $f : E \to R$ with respect to a partition $P=\{y_0,\ldots,y_n\}$ of the interval $[l, u]$ is given as

$$L(f,P)=\sum_{i=1}^{n} y_i^* m(\{x\in E \mid y_{i-1}\le f(x)\le y_i\}),$$

where $y_i^* \in [y_{i-1}, y_i]$ for all $i = 1,\ldots,n$ and f is a bounded measurable function over a bounded measurable set $E \subset R$. This is the new ways to count rectangles, the y_i^* is the height of the rectangle and the $m(\{x\in E \mid y_{i-1}\le f(x)\le y_i\})$ serves as the base of the rectangle. The definition of the actual Lebesgue integral is virtually identical to that of the Riemann integral.

Definition 4.3.2: A bonded measurable function $f : E \to R$ is Lebesgue integrable on E if there is a number $L \in R$ such that given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|L(P,f)-L| < \varepsilon$ whenever $\|P\| < \delta$, where L is known as the Lebesgue integral of f on E and is denoted by $\int_E f dm$.

Theorem 4.3.3: A bounded measurable function f is Lebesgue integrable on a bounded measurable set E if and only if, given $\varepsilon > 0$, there exists simple functions $\overline{f}$ and $\underline{f}$ such that

$$\underline{f} \le f \le \overline{f},$$

and

$$\int_E \overline{f} - \int_E \underline{f} < \varepsilon.$$

Proof: If f is a bounded measurable function on a bounded measurable set E, then f is a measurable on E. Furthermore,

$$\int_E f dm = \sup\left\{ \int_E \underline{f} dm \mid \underline{f} \text{ is simple and } \underline{f} \le f \right\}.$$

$$= \inf\left\{ \int_E \bar{f}dm \mid \bar{f} \text{ is simple and } \bar{f} \geq f \right\}. \square$$

Theorem 4.3.4 (Bounded Convergence Theorem): Suppose $\{f_n\}$ is a sequence of real-valued measurable functions on $[a, b]$ for which there exists a positive constant M such that $|f_n(x)| \leq M$ for all $n \in N$, and all $x \in [a,b]$. If

$$\lim_{n\to\infty} f_n(x) = f(x),$$

then is integrable on $[a, b]$ and

$$\int_{[a,b]} fdm = \lim_{n\to\infty} \int_{[a,b]} f_n dm.$$

Proof: Since $f_n \to f$, f is measurable and thus Lebesgue integrable. Let

$$E = \{x \in [a,b] : f_n(x) \text{ does not convergence to } f(x)\}.$$

The function g, g_n, $n \in N$ is defined on $[a, b]$,

$$g_n(x) = \begin{cases} f_n(x), & x \in [a,b] \setminus E, \\ 0, & x \in E, \end{cases}$$

and

$$g(x) = \begin{cases} f(x), & x \in [a,b] \setminus E, \\ 0, & x \in E. \end{cases}$$

Since $m(E) = 0$, $g_n = f_n$ and $g = f$. Therefore,

$$\int_a^b g_n dm = \int_a^b f_n dm$$

and

$$\int_a^b g\, dm = \int_a^b f\, dm.$$

Furthermore $g_n(x) \to g(x)$ for all $x \in [a,b]$. Let $\varepsilon > 0$ be given. For $m \in N$, set

$$E_m = \{x \in [a,b] : |g(x) - g_n(x)| < \varepsilon \text{ for all } n \geq m\}.$$

Then $E_1 \subset E_2 \subset \cdots$ with $\bigcup_{m=1}^{\infty} E_m = [a,b]$. Therefore,

$$\bigcap_{m=1}^{\infty} E_m' = \emptyset.$$

Hence, $E'_m = [a, b] \setminus E_m$. We have $\lim_{m\to\infty} m(E'_m) = 0$. Choose $m \in N$ such that $m(E'_m) < \varepsilon$. Then $| g(x) - g_n(x) | < \varepsilon$ for all $n \geq m$ and all $x \in E_m$. Therefore,

$$\left|\int_a^b f dm - \int_a^b f_n dm\right| = \left|\int_a^b g dm - \int_a^b g_n dm\right| \leq \int_{[a,b]} | g - g_n | dm$$

$$= \int_{E_m} | g - g_n | dm + \int_{E'_m} | g - g_n | dm$$

$$< \varepsilon m(E_m) + 2M m(E'_m) < \varepsilon[b - a + 2M].$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{n\to\infty} \int_{[a,b]} f_n dm = \int_{[a,b]} f dm. \ \square$$

Theorem 4.3.5 (**Fatou's lemma):** If $\{f_n\}$ is a sequence of nonnegative measurable functions on a measurable set A, and $\lim_{n\to\infty} f_n(x) = f(x)$ on A, then

$$\int_A f dm \leq \underline{\lim}_{n\to\infty} \int_A f_n dm.$$

Proof: Suppose that the set A bounded. For each $k \in N$, let

$$h_n(x) = \min\{f_n(x), k\} \qquad \text{and} \qquad h(x) = \min\{f(x), k\}.$$

Then for each $k \in N$, the sequence $\{h_n\}$ converges to h on A. since $| h_n(x) | \leq k$ for all $x \in A$, by the bounded convergence theorem.

$$\int_A \min\{f, k\} dm \leq \underline{\lim}_{n\to\infty} \int_A \min\{f_n, k\} dm \leq \underline{\lim}_{n\to\infty} \int_A f_n dm.$$

Since the above holds for each $k \in N$,

$$\int_A f dm = \underline{\lim}_{k\to\infty} \int_A \min\{f, k\} dm \leq \underline{\lim}_{n\to\infty} \int_A f_n dm.$$

If A is unbounded, then by the above for each $k \in N$,

$$\int_{A\cap[-k,k]} f\, dm \leq \underline{\lim}_{n\to\infty} \int_{A\cap[-k,k]} f_n dm \leq \underline{\lim}_{n\to\infty} \int_A f_n dm. \ \square$$

Remark: Fotou's lemma is often to used to prove that the limit function f of a convergence sequence of nonnegative Lebesgue integrals functions is Lebesgue integrable. For if $\underline{\lim}\int_A f_n dm < \infty$ and if $f_n \to f$ on A with $f_n \geq 0$ for all n, then by the Fatou's lemma, $\int_A f_n dm < \infty$. Thus f is integrable on A.

Theorem 4.3.6 (Lebesgue's Dominated Convergence Theorem): Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set A such that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for all $x \in A$. Suppose there exists a nonnegative integrable functions g on A such that $|f_n(x)| \leq g(x)$ for all $x \in A$. Then f is integrable on A and

$$\int_A fdm = \lim_{n\to\infty} \int_A f_n dm.$$

Proof: Since g is integrable on A. the functions f and f_n also has finite Lebesgue integrals. By redefining all the f_n, $n \in N$, on a set of measure zero is necessary, without loss of generality assume that $|f_n(x)| \leq g(x)$ for all $x \in A$. Consider the sequence $\{g + f_n\} \geq 0$ on the set A, by the Fotou's lemma,

$$\int_A (f+g)dm = \underline{\lim}_{n\to\infty} \int_A (f_n + g)dm \leq \underline{\lim}_{n\to\infty} \int_A (f_n + g)dm$$

$$= \int_A gdm + \underline{\lim}_{n\to\infty} \int_A f_n dm.$$

Therefore,

$$\int_A fdm \leq \underline{\lim}_{n\to\infty} \int_A f_n dm.$$

Similarly, by applying Fatou's lemma to the sequence $\{g + f_n\}$ to obtain

$$\int_A (f-g)dm \leq \underline{\lim}_{n\to\infty} \int_A (g - f_n)dm = \int_A gdm + \underline{\lim}_{n\to\infty} \int_A -f_n dm.$$

But

$$\underline{\lim}_{n\to\infty} \int_A -f_n dm = -\overline{\lim}_{n\to\infty} \int_A f_n dm.$$

Therefore,

$$\int_A fdm \geq \overline{\lim}_{n\to\infty} \int_A f_n dm. \ \square$$

Remark: The hypothesis that there exists an integrable function g satisfying $|f_n| \leq g$ for all $x \in [a,b]$ is required in the proof in order to subtract $\int g dm$ in the above inequalities. This is not possible if $\int g dm = \infty$.

Theorem 4.3.7: Let f and f_n, $n \in N$, be Riemann integrable functions on $[a, b]$ with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$. Suppose there exists a positive constant M such that $|f_n(x)| \leq M$ for all $x \in [a,b]$ and all $n \in N$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Discussion: Recalls that $\{f_n\}$ is uniformly convergent, if given $\varepsilon > 0$, there exists a natural number N such that $|f_n - f_n(x)| < \varepsilon$ whenever $n > N$, for all $x \in [a,b]$. The hypotheses placed on $\{f_n\}$ in order for the Lebesgue Dominated Convergence Theorem to hold are much less stringent than requiring $\{f_n\}$ converging uniformly. Thus, we can expect that the classes of Lebesgue integrable functions are better limits properties than the class of the Riemann integrable functions.

4.4 Properties of the Lebesgue Integral

The properties of the Lebesgue integral illustrate some of the techniques of Lebesgue integration.

Theorem 4.4.1: Let f and g be bounded measurable functions on a bounded measurable set E.

I. Monotonicity: If $f \leq g$, then $\int_E f\, dm \leq \int_E g\, dm$.

II. Linearity:

$$\int_E (f+g)\, dm = \int_E f\, dm + \int_E g\, dm,$$

and

$$\int_E cf\, dm = c\int_E f\, dm \ for\ c \in R.$$

III. For any number l, $u \in R$ such that $l \leq f \leq u$, it follows that $l.m(A) \leq \int_E f\, dm \leq u.m(E)$.

IV. $|\int_E f\, dm| \leq \int_E |f|\, dm$.

V. If A and B are disjoint bounded measurable sets and $f: A \cup B \to R$ is a bounded measurable function, then

$$\int_{A\cup B} f\, dm = \int_A f\, dm + \int_B f\, dm.$$

VI. Countable Additivity: if $E = E_i$ where the E_n are pairwise disjoint bounded measurable sets, then

$$\int_E f\, dm = \sum_{n=1}^{\infty} \int_{n_i} f\, dm.$$

4.5 The General Lebesgue Integral

Suppose A is a bounded measurable subset of R, and that f is a nonnegative measurable function defined on A. For each $n \in N$,

$$f_n(x) = \min\{f(x), n\} = f(x), f(x) \leq n,\ \ n, f(x) > n.$$

Then $\{f_n\}$ is a sequence of nonnegative bounded measurable functions defined on A, with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$. Furthermore, if $m > n$, then

$$f_n(x) \leq f_m(x) \leq f(x)$$

for all $x \in A$ then the sequence

$$\left\{\int_A f_n d\lambda\right\}_{n=1}^{\infty}$$

is monotone increasing, and it converges either to the real number or diverges to ∞.

Definition 4.5.1:

i) Let f be a nonnegative function defined on a bounded measurable subset A of R. The Lebesgue integral of f over A, denoted $\int_A f dm$, is defined by

$$\int_A f dm = \lim_{n\to\infty} \int_A f_n dm = \sup_n \int_A \min\{f,n\} dm .$$

ii) If A is an unbounded measurable of R and f is a nonnegative measurable function on A, the Lebesgue integral of f over A, denoted $\int_A f dm$, is defined by

$$\int_A f dm = \lim_{n\to\infty} \int_{A\cap[-n,n]} f dm .$$

The sequence of $\left\{ \int_{A\cap[-n,n]} f dm \right\}_{n\in N}$ is also monotone increasing, and thus converges either to a nonnegative real number or diverges to ∞.

Definition 4.5.2: A nonnegative measurable function f defined on a measurable-subset A of R is said to be Lebesgue integrable on A if $\int_A f dm < \infty$.

Theorem 4.5.3: Let f, g be nonnegative measurable functions defined on a measurable set A. Then

i) $\int_A (f+g) dm = \int_A f dm + \int_A g dm$ and $\int_A c f dm = c\int_A f dm$ for all $c > 0$.

ii) If A_1, A_2 are disjoint measurable subsets of A, then,

$$\int_{A_1\cup A_2} f dm = \int_{A_1} f dm + \int_{A_2} f dm .$$

iii) If $f \le g$ on A, then $\int_A f dm \le \int_A g dm$ with equality if $f = g$ on A.

Proof: i) Suppose the set A is bounded, let $h=f+g$. Since

$$\min\{f(x)+g(x),n\} \le \min\{f(x),n\} + \min\{g(x),n\} \le \min\{f(x)+g(x),2n\} .$$

We have $h_n \le f_n + g_n \le h_{2n}$ for all $n \in N$. As a consequence,

$$\int_A h_n dm \le \int_A f_n dm + \int_A g_n dm \le \int_A h_{2n} dm .$$

Suppose f,g are integrable on A. Then,

$$\lim_{n\to\infty}\left(\int_A f_n dm + \int_A g_n dm\right) = \lim_{n\to\infty}\int_A f_n dm + \lim_{n\to\infty}\int_A g_n dm = \int_A f dm + \int_A g dm .$$

Therefore, since

$$\lim_{n\to\infty}\int_A h_n dm = \lim_{n\to\infty}\int_A h_{2n} dm = \int_A (f+g) dm .$$

If one or the both $\left\{\int_A f_n dm\right\}, \left\{\int_A g_n dm\right\}$ diverges to ∞, then so theirs sum. We obtain $\int_A (f+g)dm = \infty$. If A is unbounded, then by the above for each $n \in N$,

$$\int_{A\cap[-n,n]} (f+g)dm = \int_A f dm + \int_A g dm . \square$$

Example 4.5.4: Let the function $f(x) = 1/\sqrt{x}$ defined on (0, 1). Then for each $n \in N$,

$$f_n(x) = \min\{f(x), n\} = \begin{cases} n, & 0 < x < 1/n^2, \\ 1/\sqrt{x}, & 1/n^2 \le x \le 1. \end{cases}$$

Therefore, $\int_0^1 f_n dm = \int_0^{1/n^2} n dx + \int_{1/n^2}^1 \frac{1}{\sqrt{x}} dx = \frac{1}{n} + \left(2 - \frac{2}{n}\right) = 2 - \frac{1}{n}$. As a consequence,

$\int_{(0,1)} f dm = \lim_{n\to\infty}\int_{(0,1)} f_n dm = \lim_{n\to\infty}\left(2 - \frac{1}{n}\right) = 2$. This answers corresponds t=0 improper Riemann integral of the function f. This will always be the case for nonnegative functions for which the improper Riemann integral exists.

Example 4.5.5: Let the function $f(x) = 1/\sqrt{x}$ defined on (0, 1). Then for each $n \in N$,

$$f_n(x) = \min\{f(x), n\} = \begin{cases} n, & 0 < x < 1/n^2, \\ 1/\sqrt{x}, & 1/n^2 \le x \le 1. \end{cases}$$

Therefore, $\int_0^1 f_n dm = \int_0^{1/n^2} n dx + \int_{1/n^2}^1 \frac{1}{\sqrt{x}} dx = \frac{1}{n} + \left(2 - \frac{2}{n}\right) = 2 - \frac{1}{n}$. As a consequence,

$\int_{(0,1)} f dm = \lim_{n\to\infty}\int_{(0,1)} f_n dm = \lim_{n\to\infty}\left(2 - \frac{1}{n}\right) = 2$. This answers corresponds t=0 improper Riemann integral of the function f. This will always be the case for nonnegative functions for which the improper Riemann integral exists.

CHAPTER 5

DISCUSSIONS, CONCLUSIONS AND SUGGESTIONS

5.1 Differentiation between the Riemann and Lebesgue Integral

The Riemann integral are defined as follows:

- subdivide the domain of the function (usually a closed, bounded interval) into finitely many subintervals (the partition)
- construct a simple function that has a constant value on each of the subintervals of the partition (the Upper and Lower sums)
- take the limit of these simple functions as to add more and more points to the partition

If the limit exists, it is called the Riemann integral and the function is called Riemann integrable. Now i will take, in a manner of speaking, the "opposite" approach:

- subdivide the range of the function into finitely many pieces
- construct a simple function by taking a function whose values are those finitely many numbers
- take the limit of these simple functions as to add more and more points in the range of the original function

If the limit exists, it is called the Lebesgue integral and the function is called Lebesgue integrable. To define this new concept, we use several steps:

1. Define the Lebesgue Integral for "simple functions".
2. Define the Lebesgue integral for bounded functions over sets of finite measure.
3. Extend the Lebesgue integral to positive functions (that are not necessarily bounded).
4. Define the general Lebesgue integral.

5.2 Relation between the Riemann Integral and The Lebesgue Integral

The Riemann integral and the Lebesgue have much more relation with each other. We will show that the basic properties of the Riemann integral of a real-value function and to relate it to the Lebesgue integral.

The Riemann integral of a bounded real-valued function $f:[a,b]\rightarrow R$, and f can take positive or negative value, but it is essential that f be a bounded function and the domain of f be a compact interval. We have the Riemann integrable which is $\int_{-a}^{b} f = \int_{a}^{-b} f$, where rather than $\int_{a}^{b} f(x)dx$, so that this integral is not confuse with the Lebesgue integral, this notation will reserve for the Lebesgue integral.

For every continuous function $f \in R[a,b]$ is Riemann integrable. Lebesgue integrable is also at the function $f \in R[a,b]$. Both Riemann and Lebesgue integral are agree $\int_{a}^{b} f(x)dx = \int_{-a}^{b} f = \int_{a}^{-b} f$ on every $f \in R[a,b]$. The Riemann integrable makes sense only for function f that is define on a bounded and compact interval. Continuous functions are Riemann integrable and their Riemann and Lebesgue integrals coincide.

Let f be a nonnegative function in $R[a,\infty)$. If f is an improperly Riemann integrable then f belongs to the Lebesgue space $L[a,\infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{T\to\infty} I_T(f).$$

For the this situation, a continuous function f defined on $[a,\infty)$, and $f(x)\geq 0, x\geq a$. Riemann integral can be form at every $T>a$, then f is restricts to a nonnegative function in $C[a,T]$. These integrals are numbers that depends on T and denote them by $I_T(f)$. $I_T(f)$ is an increasing function on T since f is nonnegative. f is said to be improperly integrable if the partial integral $I_T(f)$ remain bounded. Improper integral of f is define by

$$I(f) = \lim_{T\to\infty} I_T(f).$$

The second type of the improper integral is defined on bounded interval like $(a,b],[a,b)$, or (a,b).but which is unbounded in their domain. Such as

$$f(x) = \frac{1}{\sqrt{x}}, \quad 0<x<1.$$

Consider a continuous negative function f defined on an interval $(a, b]$ but it unbounded near the left endpoint. That is, for every positive numbers $0<\in<b-a$, the function is Riemann integrable where the restriction of f to the compact subinterval $[a+\varepsilon, b]$. We denote the Riemann integral by $I^{\in}(f)$. When ε decrease to a, the integrals $I^{\in}(f)$ increase. f is said to be improperly Riemann integrable if $I^{\in}(f)$ remain bounded and then it belongs to Lebesgue integral space $L(a, b]$ and $\int_{(a,b]} f(x)dx = \lim_{\in\to a^+} I^{\in}(f)$.

Let f be a bounded real-value function defined on a compact interval $[a, b]$. Then f is Riemann integrable if the set $D = \{x\in[a,b] : f \text{ is not continuous at } x\}$ of all discontinuity points of f is a set of Lebesgue measure zero. Every bounded Riemann integrable function defines on $[a, b]$ is Lebesgue integral, and the two integral are the same.

5.3 Comparison to The Riemann Integral

The definition of the Lebesgue integral is very similar to that of the Riemann integral, except that in the Lebesgue theory we use measurable partitions rather than point partition. If $P=\{x_0,x_1,....,x_n\}$ is a partition on $[a,\ b]$, then

$$P^*=\{[x_0,x_1]\},\bigcup\{[x_{k-1},x_k]\}_{k=2}^{n}$$

is a measurable partition of $[a,\ b]$. Furthermore, if f is a bounded real-valued function on $[a,\ b]$, then,

$$L(f,P)\leq L_l(f,P^*) \qquad \text{and} \qquad U(f,P)\geq U_l(f,P^*).$$

Therefore, the lower Riemann integral of f satisfies

$$\underline{\int_a^b} f=\sup\{L(f,P): P \text{ is a partition of } [a\ b]\}$$

$$\leq \sup\{L_l(f,L): L \text{ is a measurable partition of } [a,\ b]\}.$$

Similarly, for the upper Riemann integral of f we have

$$\overline{\int_a^b} f=\inf\{U_l(f,L): L \text{ is a measurable of } [a,\ b]\}.$$

If f is a Riemann integrable on $[a,\ b]$, then the upper and lower Riemann integrals of f are equal, and thus

$$\int_a^b f(x)dx\leq \sup_L L_l(f,L)\leq \inf_L U_l(f,L)\leq \int_a^b f(x)dx$$

where the supremum and infimum are taken over all measurable functions L of $[a,\ b]$. With the Theorem 4.3.6, the Lebesgue's Dominated Convergence Theorem and Theorem 4.3.7, the Convergence of the Riemann integrals are stated explicitly related the Riemann and Lebesgue integrals was :

If f is Riemann integrable on $[a,\ b]$, then f is Lebesgue integrable on $[a,\ b]$, and

$$\int_a^b f(x)dx=\int_{[a,b]} fdm.$$

We have seen that the converse theorem is not true. Thus, not only does the class of Lebesgue integrable functions have better limits properties, but it is also larger than the class Riemann integrable functions. There is an example to re-examine of the function below.

Example : Consider the sequence of functions $\{f_n\}$ over the interval E=[0, 1].

$$f(x)=\begin{cases}2^n, & \text{if } \frac{1}{2^n}\le x\le\frac{1}{2^{n-1}},\\ 0, & \text{ortherwise.}\end{cases}$$

The limits function of this sequence is simply $f = 0$. In this example, each function in the sequence is Rieamann integrable, as in the limit function. However, the limit of the sequence of Riemann integral is not equal to the Riemann integral of the limit of the sequence. That is,

$$\lim_{n\to\infty}\int_0^1 f_n(x)=1\neq 0=\int_0^1 \lim_{n\to\infty} f_n(x)dx \ .$$

Then, it is also the case that

$$\lim_{n\to\infty}\int_{[0,1]} f_n dm=1\neq 0=\int_{[0,1]}\lim_{n\to\infty} f_n(x)dx \, .$$

Thus, we have to show solve the function f_n by measurable function. There are three cases to consider, corresponding to three possible choices for the real value number r. They are

a) $r>2^n$: The set $\{x\in E\,|\,f_n(x)>r\}$ is a null set and therefore measurable.

b) $0\le r<2^n$: The set $\{x\in E\,|\,f_n(x)>r\}$ is the closed interval $\left[\frac{1}{2^n},\frac{1}{2^{n-1}}\right]$ and is measurable.

c) $r>0$: The set $\{x\in E\,|\,f_n(x)>r\}$ is the entire interval E and is therefore measurable.

Thus, each f_n is a measurable function. This example could show that although the Lebesgue integral is superior to the Riemann integral insofar as the size of the class of integrable functions and the limit properties of these functions, there are still functions which defy Lebesgue integration.

Discusssion: Definition of the Riemann integration is simple and clearly motivated as a measure of area, it is well suited to formulating physical laws and performing computations and it articulates the relationship between integration and differentiation through the fundamental theorem of calculus. The Riemann integrable functions is not wide enough, with consider the following observation:

1. In example The function defined on R by

$$f(x)=\begin{cases}1, & x\in Q,\\ 0, & x\in Q^c,\end{cases}$$

is not Riemann integrable on any bounded interval, though the function is almost constant.

2. The set of Riemann integrable functions $R[a, b]$ is not closed under pointwise convergence. It is not even closed under monotonic pointwise convergence, let $(f_n = n\in N)$ be the sequence of functions defined on [0, 1] by

$$f_n(x)=\begin{cases}1, & x=p/q\in Q\cap[0,1], q\le n,\\ 0, & \text{ortherwise},\end{cases}$$

where p and q have no common factors, $(f_n\in R(0,1)$ because f_n is continuous except at a finite number of points. Clearly, $f_n(x)$ increases with n and tends to the functions $f(x)$ as $n\to\infty$.

The main difference between the Riemann Integral and the Lebesgue Integral is that the Riemann Integral is using over a partitioning on an interval and the Lebesgue Integral is over a partition of a set. If $A\subset R$, than the measurable function of A is a finite collection $\{A_n\}$ of pairwise disjoint measurable subsets of A such that

$$A=\bigcup_{i=1}^{n} A_i .$$

If the partition $P=\{x_0, x_1, \ldots, x_n\}$ of the bounded interval $[a, b]$, then set $A_1=[x_1, x_2]$ and $A_n=[x_{n-1}, x_n]$ for n=2, then we can easily see that the collection $\{A_n\}$ is a measurable functions of $[a, b]$ and does not need as an intervals.

Even through both the Riemann Integral and Lebesgue Integrals are related to supremums and infimums of sums over partitions. In facts if f is bounded real-valued function on $[a, b]$ and $\{A_n\}$ is a measurable partitions of $[a, b]$ then similar to the Riemann upper and lower sums. The Lebesgues sums are defined as

$$L(f, A) = \sum_{i=1}^{n} m_i \lambda(A_i) \qquad \text{and} \qquad U(f, A) = \sum_{i=1}^{n} M_i \lambda(A_i),$$

where

$$m_i = \inf\{f(t) : t \in A_i\} \qquad \text{and} \qquad M_i = \sup\{f(t) : t \in A_i\}.$$

Lebesgue can apply to some more spaces in where the Riemann cannot, it is because the Lebesgue Integaral uses a more concept of measure to address the length of the domain partition. This allows more flexibility for integration therefore the construction of the integral can be done with respects to sets and measure on those sets. The Riemann and Lebesgue can be distinguishing such as following:

i. $\int_a^b f(t)dt$ is use when the Riemann integral of f does exists.

ii. $\int_a^b f(t)d\lambda(t)$ is apply for the Lebesgue Integral.

Contrast to the Riemann condition, Lebesgue integrable does not have to bounded and continuous anywhere while the weakness of the Riemann Integral can only be applied to bounded real-valued functions. Lebesgue Integral also performs better in the limits functions while the Riemann Integral is difficult to describe in the limit processes.

5.4 Conclusions

From the results showing in chapter 4, although the Riemann integration are complicated enough since being defined as a limit of upper and lower sums, Riemann integrable functions are almost continuous functions, and it can only be taken over or unions of intervals so it cannot be used to integrate the abstracts of sets. And thus, Riemann faces difficulties to extend to other abstracts sets, for example, from N to R. So the most appropriate integral to take place is to solve the weakness of the Riemann. Riemann integral and Lebesgue integral are much more similar. But the Lebesgue integral is more flexible and useful than the Riemann integral. The fruits of this theory are now available in a sequence of powerful results which include the monotone convergence theorem, Fatou's lemma, the dominated convergence theorem and the bounded convergence theorem. They all point the superior behavior of the Lebesgue Integral compare to the Riemann integral insofar as the size of the class of integrable functions and the limits properties.

5.5 Suggestions

This is a very interesting topic under the real analysis course, it is good idea that to teach both the Riemann integral and Lebesgue integral in undergraduate level to consider in the syllabus of the mathematics course. It can be used as a resource for self-study by those students who want to deeper understanding of integration. Anyone intending to continue in graduate study or further master in pure or applied mathematics can consider of these topic.

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BIODATA

Name	:	Chow Lee Kum
Address	:	49, Kampung Baru, 08300 Gurun, Kedah.
Telephone	:	017-4307749
Email	:	kumisme@hotmail.com
Date of Birth	:	04 October 1986
Place of Birth	:	Sungai Petani, Kedah
Nationality	:	Malaysian
Race	:	Chinese
Gender	:	Female
Religion	:	Buddhist
Education	:	University Malaysia Terengganu (2006-2009) SMK Aman Jaya, Sugai Lalang (2004-2005) SMK Gurun, Gurun (1999-2003) SRJK (C) Choong Hwa, Gurun (1993-1998