Journal of Sustainability Science and Management
Volume 6 *Number* 1, June 2011: 44-50

ISSN: 1823-8556
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NUMERICAL SIMULATION DYNAMICAL MODEL OF THREE-SPECIES FOOD CHAIN WITH LOTKA-VOLTERRA LINEAR FUNCTIONAL RESPONSE

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Abstract: In this paper, we study an ecological model with a tritrophic food chain with a classical Lotka-Volterra functional response. There are three equilibrium points of the system. In the parameter space, there are passages from instability to stability, which are called Hopf bifurcation points. For the equilibrium point, it is possible to find bifurcation points analytically and to prove that the system has periodic solutions around these points. Furthermore, the dynamical behaviours of this model are investigated. It has been found that the extinction of the predator depends on the prey-predator parameter in the equation. The dynamical behaviour is found to be very sensitive to the parameter values and the initial condition as well as the parameters of the practical life. Computer simulations are carried out to explain the analytical findings.

KEYWORDS: food chain model, modified Lotka-Volterra model, Hopf bifurcation

Introduction

The current technological advance has made it possible for humans to disturb the environmental balance in nature that may cause immense damages, such as species extinction or starvation. Therefore, understanding the behaviour of the interaction between the species may help biologists and other related parties to prevent those events from happening.

The real interaction of prey-predator in nature is complex and comprises both interspecies and external environmental factors. Therefore, several simplifications are usually assumed so that a basic model can be constructed and then developed or modified to approach the real system.

One of the simplest dynamical models to describe the interaction between two interacting species, namely one prey and one predator, is the classical Lotka-Volterra (Chauvet *et al.,* 2002) equation which can be stated as

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 xy \\ \frac{dy}{dt} &= -a_2 y + b_2 xy \end{aligned}\right\} \qquad (1)$$

where x is a prey, y is a predator, the predator y preys on x, a_1 is the prey growth rate in the absence of the predators, b_1 is the capture rate of prey per predator, b_2 is the rate at which each predator converts captured prey into predator births and a_2 is the constant rate at which death occurs in the absence of prey. They showed that ditrophic food chains (i.e. prey-predator systems) permanently oscillate for any initial condition if the prey growth rate is constant and the predator functional response is linear.

In this paper, we completely characterise the qualitative behaviour of a linear three-species food chain where the dynamics are given by classic (nonlogistic) Lotka-Volterra-type equations. We study a more basic nonlogistic system that is the direct generalisation of the classic Lotka-Volterra equations.

Received: 20 October 2010 / Accepted: 09 January 2011

Model System

The classical food chain models with only two trophic levels are shown to be insufficient to produce realistic dynamics (Chauvet *et al.,* 2002; Hsu *et al.,* 2003; Freedman and Waltman, 1977; Hastings and Powell, 1991; Klebanoff and Hastings, 1994; Mada *et al.,* 2011). Therefore, in this paper, by modifying the classical Lotka-Volterra model, we analyse and simulate the dynamics of a three-species food chain interaction. With non-dimensionalisation, the system of three-species food chain can be written as

$$\left.\begin{aligned} \frac{dx}{dt} &= (a_1 - b_1 y)x \\ \frac{dy}{dt} &= (-a_2 + b_2 x - c_1 z)y \\ \frac{dz}{dt} &= (-a_3 + c_2 y)z \end{aligned}\right\} \qquad (2)$$

where x, y, and z denote the non-dimensional population density of the prey, predator, and top predator respectively. The predator y preys on x and the predator z preys on y. Furthermore a_1, a_2, a_3, b_1, b_2, c_1, and c_2 are the intrinsic growth rate of the prey, the death rate of the predator, the death rate of the top predator, predation rate of the predator, the conversion rate, predation rate of the top predator, and the conversion rate respectively.

Equilibrium Point Analysis

According to Hilborn (1994) and May (2001), the equilibrium points of (2) denoted by , are the zeros of its nonlinear algebraic system which can be written as

$$\left.\begin{aligned} (a_1 - b_1 y)x &= 0 \\ (-a_2 + b_2 x - c_1 z)y &= 0 \\ (-a_3 + c_2 y)z &= 0 \end{aligned}\right\} \qquad (3)$$

By considering the positivity of the parameters and the unknowns, we have two positive equilibrium points given by E_0 (0,0,0), and E_1 $(x_1, y_1, 0)$ with

$$x_1 = a_2 / b_2, \text{ and } y_1 = a_1 / b_1$$

and one negative equilibrium point E_2 $(0, y_2, z_2)$ with

$$y_2 = a_3 / c_2, \text{ and } z_2 = -a_2 / c_1.$$

Stability of Equilibrium Points

The dynamical behaviour of equilibrium points can be studied by computing the eigenvalues of the jacobian matrix J of system (2) where

$$J(\bar{x}, \bar{y}, \bar{z}) = \begin{bmatrix} a_1 - b_1\bar{y} & -b_1\bar{x} & 0 \\ b_2\bar{y} & -a_2 + b_2\bar{x} - c_1\bar{z} & -c_1\bar{y} \\ 0 & c_2\bar{z} & -a_3 + c_2\bar{y} \end{bmatrix} \qquad (4)$$

At most, there exists two non-negative equilibrium points for system (2). The existence and local stability conditions of these equilibrium points are as follows.

1. The Jacobian matrix (4) at the equilibrium point E_0 (0, 0, 0), is

$$J(0,0,0) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \qquad (5)$$

   The eigenvalues of the Jacobian matrix (5) are $\lambda_1 = a_1$, $\lambda_2 = -a_2$, and $\lambda_3 = -a_3$. Hence, the equilibrium point E_0 is a saddle point.

2. The Jacobian matrix (4) at the equilibrium point E_1 $(x_1, y_1, 0)$, is

$$J(x_1, y_1, 0) = \begin{bmatrix} 0 & -\frac{b_1 a_2}{b_2} & 0 \\ \frac{b_2 a_1}{b_1} & 0 & -\frac{c_1 a_1}{b_1} \\ 0 & 0 & -a_3 + \frac{c_2 a_1}{b_1} \end{bmatrix} \qquad (6)$$

   The eigenvalues of the Jacobian matrix (6) are

$$\left.\begin{aligned} \lambda_{1,2} &= \pm i\sqrt{a_1 a_2} \\ \lambda_3 &= \frac{a_1 c_2 - a_3 b_1}{b_1} \end{aligned}\right\} \qquad (7)$$

E_1 $(x_1, y_1, 0)$ is locally stable if

$$a_3 b_1 > a_1 c_2. \qquad (8)$$

Hopf Bifurcation Point

When we are interested to study periodic or quasi-periodic behaviour of a dynamical system, we need to consider the Hopf bifurcation point. The dynamical system generally (Hilborn, 1994; May, 2001 ; Mada *et al.,* 2011) can be written as

$$\dot{v} = F(v, \mu) \quad (9)$$

where

$$v = (x, y, z), = (a_1, a_2, a_3, b_1, b_2, c_1, c_2) \quad (10)$$

According to Hilborn (1994) and May (2001) for the system (2) which can be written in the form (9-10), if an ordered pair (v_0, μ_0) satisfied the conditions

(i) $F(v_0, \mu_0) = 0$,

(ii) $J(v, \mu)$ has two complex conjugate eigenvalues

$$\lambda_{1,2} = a(v, \mu) \pm ib(v, \mu),$$

around (v_0, μ_0),

(iii) $a(v_0, \mu_0) = 0$, $\nabla a(v_0, \mu_0) \neq 0$, $b(v_0, \mu_0) \neq 0$,

(iv) the third eigenvalues $\lambda_3(v_0, \mu_0) \neq 0$,

then (v_0, μ_0) is called a Hopf bifurcation point.

For the system (2), the equilibrium points E_0 (0, 0, 0), and E_1 (x_1, y_1,0), satisfy the condition $F(v_0, \mu_0) = 0$, and for the equilibrium point E_1 (x_1, y_1,0) we have two complex conjugate eigenvalues (7) with the real part of the eigenvalues being zero.

The last conditionis satisfied if

$$a_3 b_1 \neq a_1 c_2. \quad (11)$$

The equation (8) and (11) are satisfied if a_3 is chosen not as

$$a_{30} = \frac{a_1 c_2}{b_1}. \quad (12)$$

Hence, E_1 is stable for $a_3 < a_{30}$ and unstable for $a_3 > a_{30}$. The point (v_0, μ_0) which corresponds to $a_3 = a_{30}$, is a Hopf bifurcation point. This Hopf bifurcation states sufficient condition for the existence of periodic solutions. As one parameter is varied, the dynamics of the system change from a stable spiral to a centre and then to unstable spiral (see Table 1).

Numerical Simulation

Analytical studies always remain incomplete without numerical verification of the results. In this section, we present numerical simulation to illustrate the results obtained in previous sections. The numerical experiments are designed to show the dynamical behaviour of the system in three main different sets of parameters and initial conditions : I. The case $a_3 < a_{30}$ II. The case $a_3 = a_{30}$ III. The case $a_3 > a_{30}$. The coordinates of equilibrium points and the corresponding eigenvalues can be found in Table 1. For showing the dynamics of the system (2) change, the parameter set $\{a_1, a_2, b_1, b_2, c_1, c_2\}$= {0.5, 0.5, 0.5, 0.5, 0.5, 0.5} given as a fixed parameters and a_3 as a varied parameters. The calculation for the parameter set given Hopf bifurcation point $a_{30} = 0.5$ as a control parameter, equal to analysis result (12).

I. The case $a_3 < a_{30}$

For the case $a_3 < a_{30}$ (see Table 1) two eigenvalues for E_1 is pure imaginary initially-spiral stability corresponding with centre manifold in xy plane and one positive real eigenvalue corresponding with unstable one-dimensional invariant curve in z axes. Hence

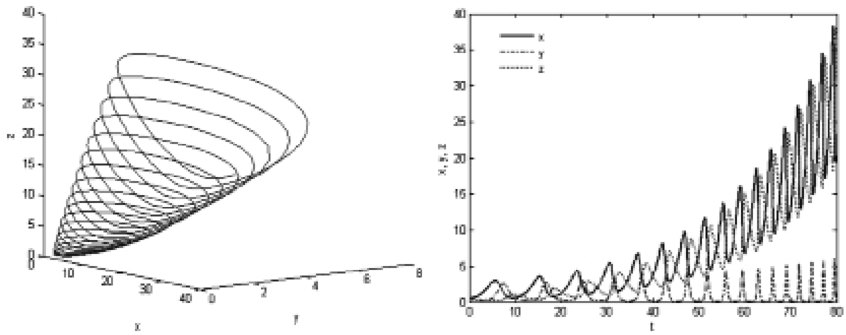
Table 1. Numerical experiment of stability equilibrium point.

The case	Parameters	Equilibrium point E_0	Equilibrium point E_1	Eigenvalues E_0	Eigenvalues E_1	Stability
$a_3 < a_{30}$	$a_3 = 0.4$	0, 0, 0	1, 1, 0	± 0.5, -0.4	± 0.5 i, 0.1	Unstable spiral
$a_3 = a_{30}$	$a_3 = 0.5$	0, 0, 0	1, 1, 0	± 0.5, -0.5	± 0.5 i, 0	Center
$a_3 > a_{30}$	$a_3 = 0.6$	0, 0, 0	1, 1, 0	± 0.5, -0.6	± 0.5 i, -0.1	Stable spiral

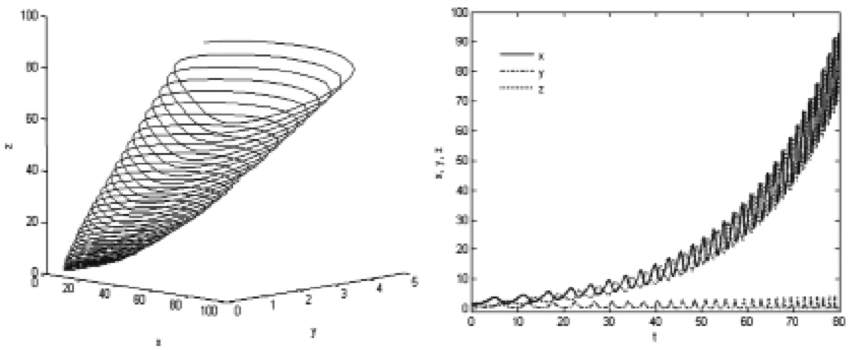
the equilibrium point E_1 is a locally unstable spiral source and E_0 are saddle points with real eigenvalues having opposite sign. In this case, the prey x and top predator z can survive, growing periodically unstable. On the other hand, predator y persists and has populations that vary periodically stable. The solutions for this case are shown in Figure 1.

II. The case $a_3 = a_{30}$

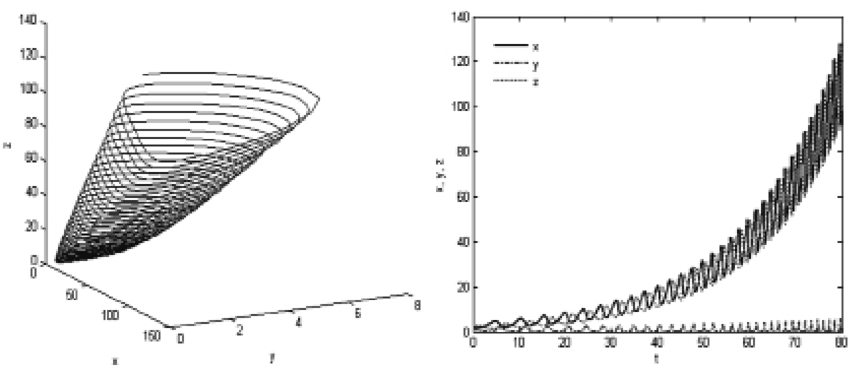
For the case $a_3 = a_{30}$ the equilibrium E_1 has three eigenvalues with zero real part corresponding with stable centre point in xy plane (see Table 1). In this case, prey x, predator y and top predator z persist and has populations that vary periodically over time with a common period as shown in Figure 2.



(a) Phase space and time series for $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$



(b) Phase space and time series for $(x_0, y_0, z_0) = (1.5, 1.5, 1.5)$

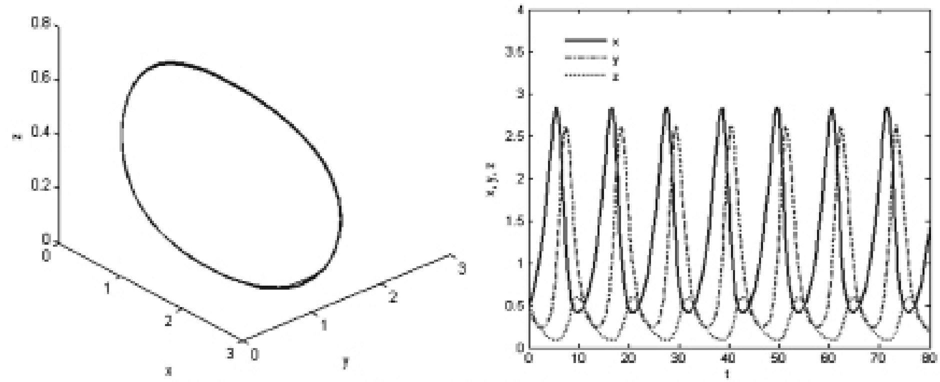


(c) Phase space and time series for $(x_0, y_0, z_0) = (2, 2, 2)$

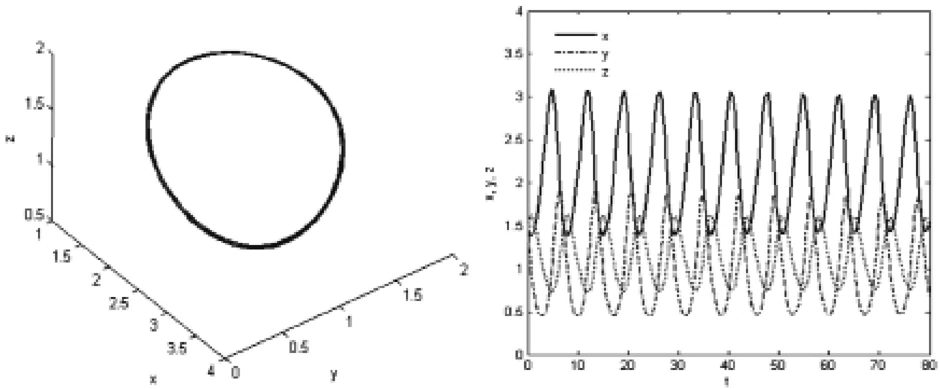
Figure 1: The solution for with $a_3 = 0.4$ and $t = 80$ for different initial condition.

III. The case $a_3 > a_{30}$

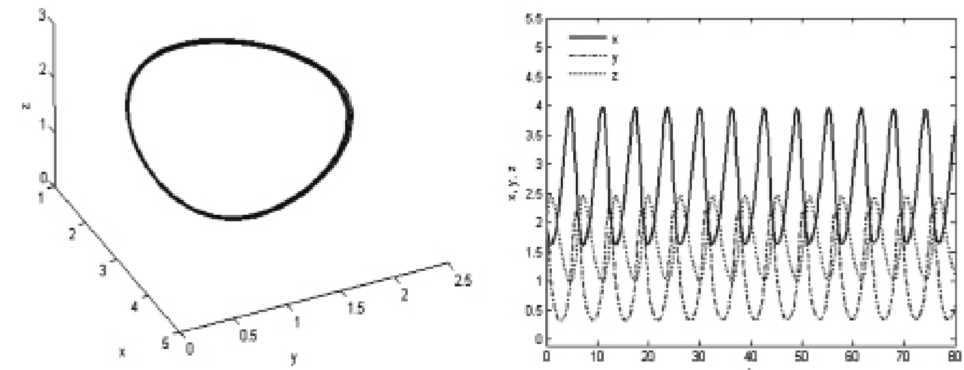
For the case $a_3 > a_{30}$ (see Table 1) two eigenvalues for E_1 are pure imaginary initially-spiral stability corresponding with centre manifold in *xy* plane and one negative real eigenvalue corresponding with stable one-dimensional invariant curve in *z* axes. Hence the equilibrium point E_1 is locally stable spiral sink and E_0 is a saddle point



(a) Phase space and time series for $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$



(b) Phase space and time series for $(x_0, y_0, z_0) = (1.5, 1.5, 1.5)$



(c) Phase space and time series for $(x_0, y_0, z_0) = (2, 2, 2)$

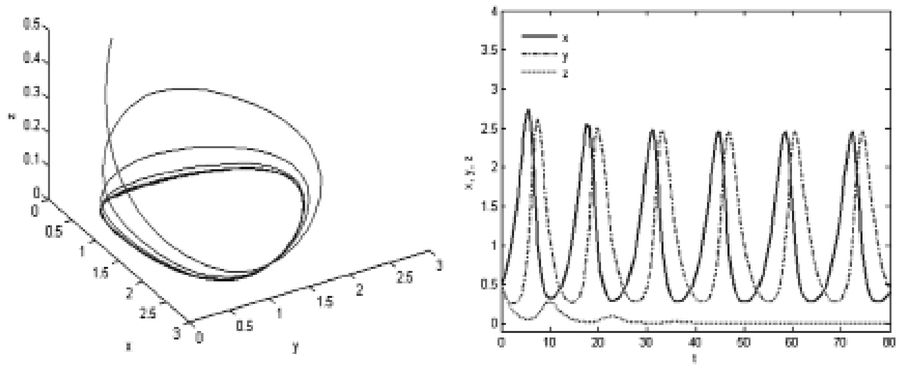
Figure 2: The solution for with $a_3 = 0.5$ and $t = 80$ for different initial condition.

with real eigenvalues having opposite sign. In this case, top predator dies. On the other hand, prey x and predator y persist and has populations that vary periodically over time with a common period. The solutions are plotted in Figure 3.

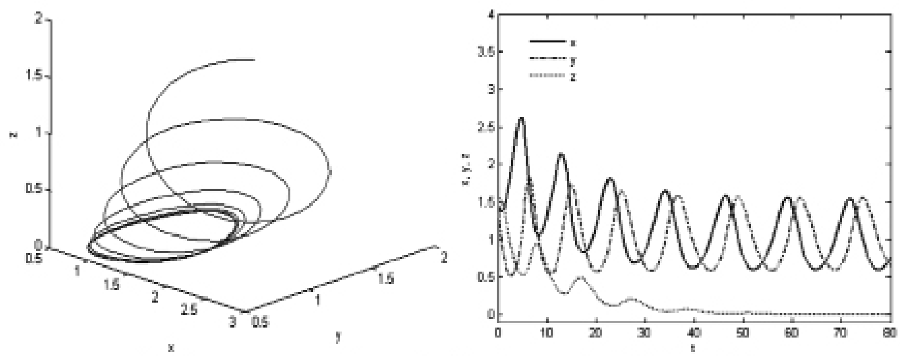
Persistence of top species z in (2) depends on the parameters a_1, a_3, b_1, and c_2. In particular, if, then species z decrease over time to die, while if, then species z survives. On the other hand, species prey x and middle predator y can persist for all conditions with stable asymptotically.

Conclusions

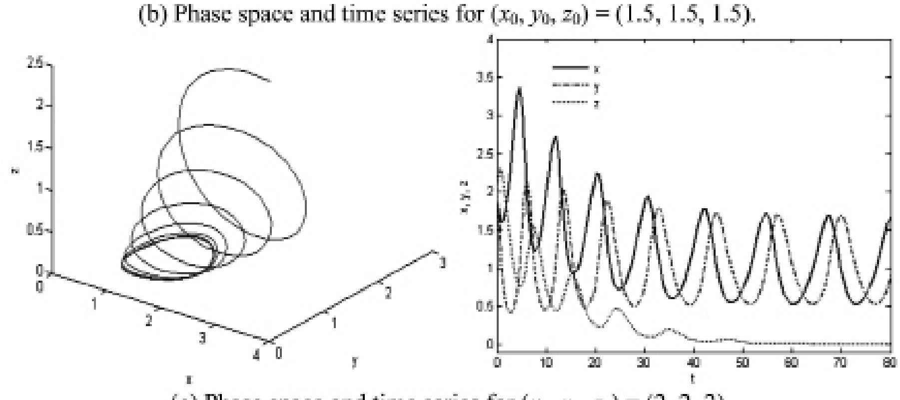
In this paper, a modified ecological model with a tritrophic food chain of a classical Lotka-Volterra



(a) Phase space and time series for $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$



(b) Phase space and time series for $(x_0, y_0, z_0) = (1.5, 1.5, 1.5)$.



(c) Phase space and time series for $(x_0, y_0, z_0) = (2, 2, 2)$

Figure 3: The solution for with $a_3 = 0.6$ and $t = 80$ for different initial conditions.

linear functional response are studied. Three-species food chain model are analysed and possible dynamical behaviour of this system is investigated at equilibrium points. It has been shown that the solutions possess Hopf bifurcations, as one parameter is varied, the dynamics of the system change from a stable to a centre to unstable. Both analytically and numerically, simulation shows that in certain regions of the parameter space, the model sensitively depends on the parameter values and initial condition.

Acknowledgement

The authors gratefully acknowledge the financial support from Ministry of Higher Education under the FRGS Vot 59146.

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