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Complete Solution To Chromatic Uniqueness Of K_4-Homeomorphs With Girth 9*

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Abstract

For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e. H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. In this paper, we completely determine the chromaticity of K_4-homeomorphs which has girth 9, and give sufficient and necessary condition for the graphs in the family to be chromatically unique.

1 Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e. H is isomorphic to G. Many families of χ-unique graphs are known (see [9, 10, 11]).

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ if the six edges of K_4 are replaced by the six paths of length a, b, c, d, e, f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 7$ has been studied by many authors (see [3, 12, 14, 15, 16]). Also the study of the chromaticity of K_4-homeomorphs with at least 2 paths of length 1 has been fulfiled (see [4, 13, 14, 22]). Recently, Shi et al. [18] studied the chromaticity of one family of K_4-homeomorphs with girth 8, i.e. $K_4(2, 3, 3, d, e, f)$. In [19], Shi has solved completely the chromaticity of K_4-homeomorphs with girth 8. As we know, only the chromaticity of such graphs with at least 2 paths of length 1 have been obtained among all the K_4-homeomorphs with girth 9. By Ren [17], the chromaticity of K_4-homeomorphs with exactly 3 paths of same length has been obtained. Recently, Catada-Ghimire and Hasni [1] investigated the chromaticity of K_4-homeomorphs with exactly 2 paths of length 2. Hence, to completely

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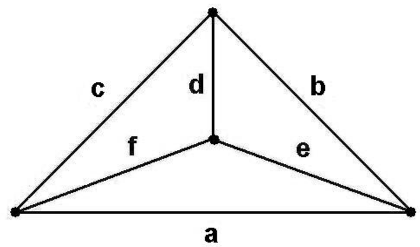

Figure 1: $K_4(a, b, c, d, e, f)$.

determine the chromaticity of K_4-homeomorph with girth 9, there are only 6 more types to be solved, that is, $K_4(1, 2, 6, d, e, f)$, $K_4(1, 3, 5, d, e, f)$, $K_4(1, 4, 4, d, e, f)$, $K_4(2, 3, 4, d, e, f)$, $K_4(1, 2, c, 3, e, 3)$ and $K_4(1, 3, c, 2, e, 3)$. The chromaticity of the graphs $K_4(2, 3, 4, d, e, f)$, $K_4(1, 4, 4, d, e, f)$ and $K_4(1, 2, 6, d, e, f)$ were solved by Karim et al. [6, 7, 8]. In this paper, to complete the study of the chromaticity of K_4-homeomorph with girth 9, we investigate the remaining types $K_4(1, 3, 5, d, e, f)$, $K_4(1, 2, c, 3, e, 3)$ and $K_4(1, 3, c, 2, e, 3)$. As by-product, we obtain the complete solution on the chromaticity of all families of K_4-homeomorphs with girth 9.

2 Preliminary Results

In this section, we give some known results used in the sequel.

LEMMA 1. Assume that G and H are χ-equivalent. Then

(1) $|V(G)| = |V(H)|$ and $|E(G) = |E(H)|$ ([9]);

(2) G and H has the same girth and same number of cycles with length equal to their girth ([21]);

(3) If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph ([2]);

(4) Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$. Then

 (i) min (a, b, c, d, e, f) = min (a', b', c', d', e', f') and the number of times that this minimum occurs in the list $\{a, b, c, d, e, f\}$ is equal to the number of times that this minimum occurs in the list $\{a', b', c', d', e', f'\}$ ([20]);

 (ii) if $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets, then $H \cong G$ ([12]).

LEMMA 2 (Ren [17]). Let $G = K_4(a, b, c, d, e, f)$ when exactly three of a, b, c, d, e, f are the same. Then G is not chromatically unique if and only if G is isomorphic to $K_4(s, s, s -$

$2,1,2,s)$ or $K_4(s,s-2,s,2s-2,1,s)$ or $K_4(t,t,1,2t,t+2,t)$ or $K_4(t,t,1,2t,t-1,t)$ or $K_4(t,t+1,t,2t+1,1,t)$ or $K_4(1,t,1,t+1,3,1)$ or $K_4(1,1,t,2,t+2,1)$, where $s \geq 3$, $t \geq 2$.

LEMMA 3 (Hasni et al. [5]). Let K_4-homeomorphs $K_4(1,3,5,d,e,f)$ and $K_4(1,3,5,d', e', f')$ be chromatically equivalent. Then

$$K_4(1,3,5,i,i+6,i+1) \sim K_4(1,3,5,i+2,i,i+5)$$

and

$$K_4(1,3,5,i,i+1,i+4) \sim K_4(1,3,5,i+2,i+3,i),$$

where $i \geq 1$.

LEMMA 4 (Karim et al. [6]). Let K_4-homeomorphs $K_4(2,3,4,d,e,f)$ and $K_4(1,3,5,d', e', f')$ be chromatically equivalent. Then

$$K_4(2,3,4,1,3,6) \sim K_4(1,3,5,4,4,2)$$

and

$$K_4(2,3,4,1,5,7) \sim K_4(1,3,5,2,8,3).$$

LEMMA 5 (Karim et al. [7]). Let K_4-homeomorphs $K_4(1,4,4,d,e,f)$ and $K_4(1,3,5,d', e', f')$ be chromatically equivalent. Then

$$K_4(1,4,4,3,5,8) \sim K_4(1,3,5,5,7,4),$$

$$K_4(1,4,4,6,3,7) \sim K_4(1,3,5,4,4,8),$$

$$K_4(1,4,4,6,3,8) \sim K_4(1,3,5,4,9,4),$$

and

$$K_4(1,4,4,6,2,6) \sim K_4(1,3,5,2,4,8).$$

LEMMA 6 (Karim et al. [8]). Let K_4-homeomorphs $K_4(1,2,6,d,e,f)$ and $K_4(1,3,5,d', e', f')$ be chromatically equivalent. Then

$$K_4(1,2,6,4,5,8) \sim K_4(1,3,5,2,6,9),$$

$$K_4(1,2,6,4,7,5) \sim K_4(1,3,5,2,8,6),$$

$$K_4(1,2,6,3,4,10) \sim K_4(1,3,5,9,2,6),$$

$$K_4(1,2,6,3,4,6) \sim K_4(1,3,5,5,6,2),$$

$$K_4(1,2,6,5,3,8) \sim K_4(1,3,5,7,2,7),$$

$$K_4(1,2,6,5,9,3) \sim K_4(1,3,5,7,8,2),$$

and

$$K_4(1,2,6,f+2,4,f) \sim K_4(1,3,5,2,f,f+4),$$

where $f \geq 4$.

LEMMA 7 (Catada-Ghimire and Hasni [1]). A K_4-homeomorphic graph with exactly two path of length two is χ-unique if and only if it is not isomorphic to

$$\begin{array}{lclcl}
K_4(1,2,2,4,1,1), & \text{or} & K_4(4,1,2,1,2,4), & \text{or} & K_4(1,s+2,2,1,2,s), \\
K_4(1,2,2,t+2,t+2,t), & \text{or} & K_4(1,2,2,t,t+1,t+3), & \text{or} & K_4(3,2,2,r,1,5), \\
K_4(1,r,2,4,2,4), & \text{or} & K_4(3,2,2,r,1,r+3), & \text{or} & K_4(r+2,2,2,1,4,r), \\
K_4(r+3,2,2,r,1,3), & \text{or} & K_4(4,2,2,1,r+2,r), & \text{or} & K_4(3,4,2,4,2,6), \\
K_4(3,4,2,4,2,8), & \text{or} & K_4(3,4,2,8,2,4), & \text{or} & K_4(7,2,2,3,4,5), \\
K_4(5,2,2,3,4,7), & \text{or} & K_4(8,2,2,3,4,6), & \text{or} & K_4(5,2,2,9,3,4), \\
K_4(5,2,2,5,3,4), & & & &
\end{array}$$

where $r \geq 3$, $s \geq 3$, $t \geq 3$.

3 Main Results

In this section, we present our main results. We now investigate the chromaticity of $K_4(1,3,5,d,e,f)$. We first obtain the following result.

LEMMA 8. Let G is of type of $K_4(1,3,5,d,e,f)$ and H is of type $K_4(1,3,c',2,e',3)$, then there is no graph satisfying $G \sim H$ unless $G \cong H$.

PROOF. Let G and H be two graphs such that $G \cong K_4(1,3,5,d,e,f)$ and $H \cong K_4(1,3,c',2,e',3)$. Let

$$\begin{aligned}
Q(K_4(a,b,c,d,e,f)) \quad = \quad & -(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+s^{a+d}+s^{b+f} \\
& +s^{c+e}+s^{a+b+e}+s^{b+d+c}+s^{a+c+f}+s^{d+e+f}.
\end{aligned}$$

Let $s = 1-\lambda$ and x is the number of edges in G. From [20], we have the chromatic polynomial of K_4-homeomorphs $K_4(a,b,c,d,e,f)$ is as follows:

$$P(K_4(a,b,c,d,e,f) = (-1)^{x-1}\frac{s}{(s-1)^2}\Big[(s^2+3s+2)+Q(K_4(a,b,c,d,e,f))-s^{x-1})\Big].$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively.

As $G \cong K_4(1,3,5,d,e,f)$ and $H \cong K_4(1,3,c',2,e',3)$, then

$$\begin{aligned}
Q(G) \quad = \quad & -\big(s+1\big)\big(s+s^3+s^5+s^d+s^e+s^f\big)+s^{d+1}+s^{f+3}+s^{e+5}+ \\
& s^{e+4}+s^{d+8}+s^{f+6}+s^{d+e+f}
\end{aligned}$$

and

$$\begin{aligned}
Q(H) \quad = \quad & -\big(s+1\big)\big(s+s^3+s^{c'}+s^2+s^{e'}+s^3\big)+s^3+s^6+s^{c'+e'}+ \\
& s^{e'+4}+s^{c'+5}+s^{c'+4}+s^{e'+5}.
\end{aligned}$$

By Lemma 1(1), we have

$$d+e+f = c'+e'. \tag{1}$$

Since $Q(G) = Q(H)$, we see that

$$Q_1(G) = -s^5 - s^6 - s^d - s^e - s^{e+1} - s^f - s^{f+1} + s^{d+8} + s^{e+4} + s^{e+5} + s^{f+3} + s^{f+6}$$

and

$$Q_1(H) = -s^2 - s^3 - s^4 - s^{c'} - s^{c'+1} - s^{e'} - s^{e'+1} + s^6 + s^{c'+4} + s^{c'+5} + s^{e'+4} + s^{e'+5}.$$

We consider the term $-s^2$ and $-s^3$ in $Q_1(H)$. Since $d+e \geq 6$ and $e+f \geq 8$, we have either $d = 3$ and $f = 2$, or $d = 2$ and $f = 3$.

Case 1. Assume that $d = 3$ and $f = 2$. We obtain the following simplification

$$Q_2(G) = -s^3 - s^6 - s^e - s^{e+1} + s^8 + s^{11} + s^{e+4} + s^{e+5},$$

$$Q_2(H) = -s^4 - s^{c'} - s^{c'+1} - s^{e'} - s^{e'+1} + s^6 + s^{c'+4} + s^{c'+5} + s^{e'+4} + s^{e'+5}.$$

Since $e \geq 6$, the term $-s^4$ is in $Q_2(H)$ but not in $Q_2(G)$, which is a contradiction.

Case 2. Assume that $d = 2$ and $f = 3$. We obtain the following simplification

$$Q_3(G) = -s^5 - s^6 - s^e - s^{e+1} + s^9 + s^{10} + s^{e+4} + s^{e+5},$$

$$Q_3(H) = -s^{c'} - s^{c'+1} - s^{e'} - s^{e'+1} + s^{c'+4} + s^{c'+5} + s^{e'+4} + s^{e'+5}.$$

We then obtain either $c' = 5$ and $e = e'$, or $c' = e$ and $e' = 5$. If $c' = 5$ and $e = e'$, we obtain $G \cong K_4(1, 3, 5, 2, e, 3)$ and $H \cong K_4(1, 3, 5, 2, e, 3)$. Hence, $G \cong H$. If $c' = e$ and $e' = 5$, we obtain $G \cong K_4(1, 3, 5, 2, e, 3)$ and $H \cong K_4(1, 3, e, 2, 5, 3)$. Hence, $G \cong H$.

So the proof is complete.

LEMMA 9. If G is of type $K_4(1, 3, 5, d, e, f)$ and H is of type $K_4(1, 2, c', 3, e', 3)$, then there are no graphs satisfying $G \sim H$ unless $G \cong H$.

PROOF. The proof is similar to Lemma 8.

LEMMA 10. If G is of type $K_4(1, 3, 5, d, e, f)$ and H is of type $K_4(2, 2, 5, d', e', f')$, then there are no graphs satisfying $G \sim H$.

PROOF. If H is of the type of $K_4(2, 2, 5, d', e', f')$, then from Lemma 7, we know that H is chromatically unique. Since $G \sim H$, we have $G \cong H$. But it is obvious that G is not isomorphic to H. This is a contradiction.

LEMMA 11. If G is of type $K_4(1, 3, 5, d, e, f)$ and H is of type $K_4(1, 2, c', 2, e', 4)$, then there are no graphs satisfying $G \sim H$.

PROOF. The proof is similar to Lemma 10.

LEMMA 12. If G is of type $K_4(1, 3, 5, d, e, f)$ and H is of type $K_4(1, 2, c', 4, e', 2)$, then there are no graphs satisfying $G \sim H$.

PROOF. The proof is similar to Lemma 10.

Now we establish the chromaticity of $K_4(1,3,5,d,e,f)$ as follows.

THEOREM 13. K_4-homeomorphs $K_4(1,3,5,d,e,f)$ with girth 9 is not χ-unique if and only if it is isomorphic to

$$\begin{array}{lclcl}
K_4(1,3,5,2,4,8), & \text{or} & K_4(1,3,5,2,8,3), & \text{or} & K_4(1,3,5,2,8,6),\\
K_4(1,3,5,4,4,8), & \text{or} & K_4(1,3,5,4,9,4), & \text{or} & K_4(1,3,5,5,6,2),\\
K_4(1,3,5,5,7,4), & \text{or} & K_4(1,3,5,7,2,7), & \text{or} & K_4(1,3,5,7,8,2),\\
K_4(1,3,5,9,2,6), & \text{or} & K_4(1,3,5,5,e,2), & \text{or} & K_4(1,3,5,e+3,2,e),\\
K_4(1,3,5,i,i+6,i+1), & \text{or} & K_4(1,3,5,i,i+1,i+4), & \text{or} & K_4(1,3,5,i+2,i,i+5),\\
K_4(1,3,5,i+2,i+3,i), & \text{or} & K_4(1,3,5,2,f,f+4), & &
\end{array}$$

where $e \geq 6$, $i \geq 1$ and $f \geq 4$.

PROOF. Let G and H be two graphs such that $G \cong K_4(1,3,5,d,e,f)$ and $H \sim G$. Since the girth of G is 9, there is at most 1 among d, e and f. Moreover, by Lemma 1(2)(3), it follows that H is a K_4-homeomorph with girth 9. So H must be one of the following 10 types.

Type 1: $K_4(1,2,6,d',e',f')$ where $d'+e' \geq 7, d'+f' \geq 6, e'+f' \geq 8$;

Type 2: $K_4(1,3,5,d',e',f')$ where $d'+e' \geq 6, d'+f' \geq 5, e'+f' \geq 8$;

Type 3: $K_4(1,4,4,d',e',f')$ where $d'+e' \geq 5, d'+f' \geq 5, e'+f' \geq 8$;

Type 4: $K_4(2,3,4,d',e',f')$ where $d'+e' \geq 6, d'+f' \geq 5, e'+f' \geq 7$;

Type 5: $K_4(2,2,5,d',e',f')$ where $d'+e' \geq 7, d'+f' \geq 5, e'+f' \geq 7$;

Type 6: $K_4(1,2,c',2,e',4)$ where $c' \geq 6, e' \geq 5$;

Type 7: $K_4(1,2,c',4,e',2)$ where $c' = e' \geq 6$;

Type 8: $K_4(1,2,c',3,e',3)$ where $c' \geq 6, e' \geq 5$;

Type 9: $K_4(1,3,c',2,e',3)$ where $c' = e' \geq 5$;

Type 10: $K_4(2,2,c',2,e',3)$ where $c' = e' \geq 5$.

If H is of Type 1, then from Lemma 6, we know that the solutions of the equation $P(G) = P(H)$ are

$$\begin{aligned}
K_4(1,2,6,4,5,8) &\sim K_4(1,3,5,2,6,9),\\
K_4(1,2,6,4,7,5) &\sim K_4(1,3,5,2,8,6),\\
K_4(1,2,6,3,4,10) &\sim K_4(1,3,5,9,2,6),\\
K_4(1,2,6,3,4,6) &\sim K_4(1,3,5,5,6,2),\\
K_4(1,2,6,5,3,8) &\sim K_4(1,3,5,7,2,7),\\
K_4(1,2,6,5,9,3) &\sim K_4(1,3,5,7,8,2),\\
K_4(1,2,6,f+2,4,f) &\sim K_4(1,3,5,2,f,f+4),
\end{aligned}$$

where $f \geq 4$.

If H is of Type 2, then from Lemma 3, we know that the solutions of the equation $P(G) = P(H)$ are

$$\begin{aligned} K_4(1,3,5,i,i+6,i+1) &\sim K_4(1,3,5,i+2,i,i+5), \\ K_4(1,3,5,i,i+1,i+4) &\sim K_4(1,3,5,i+2,i+3,i), \end{aligned}$$

where $i \geq 1$.

If H is of Type 3, then from Lemma 5, we know that the solutions of the equation $P(G) = P(H)$ are

$$\begin{aligned} K_4(1,4,4,3,5,8) &\sim K_4(1,3,5,5,7,4), \\ K_4(1,4,4,6,3,7) &\sim K_4(1,3,5,4,4,8), \\ K_4(1,4,4,6,3,8) &\sim K_4(1,3,5,4,9,4), \\ K_4(1,4,4,6,2,6) &\sim K_4(1,3,5,2,4,8). \end{aligned}$$

If H is of Type 4, then from Lemma 4, we know that the solutions of the equation $P(G) = P(H)$ are

$$\begin{aligned} K_4(2,3,4,1,3,6) &\sim K_4(1,3,5,4,4,2), \\ K_4(2,3,4,1,5,7) &\sim K_4(1,3,5,2,8,3). \end{aligned}$$

If H is of Types 5–9, then from Lemmas 8–12, we know that there is no solution of the equation $P(G) = P(H)$ unless $G \cong H$.

If H is of Type 10, then from Lemma 2, we know that H is chromatically unique. Since $G \sim H$, we have $G \cong H$. But it is obvious that G is not isomorphic to H. This is a contradiction.

This completes the proof.

The following table is to show the result of Theorem 13, that is, the solution of $P(G) = P(H)$ when $G \cong K_4(1,3,5,d,e,f)$ and $H \sim G$.

Graph H where $G \cong K_4(1,3,5,d,e,f)$ and $H \sim G$	*Solution of $P(G) = P(H)$*
Type 1: $K_4(1,2,6,d',e',f')$	From Lemma 6
Type 2: $K_4(1,3,5,d',e',f')$	From Lemma 3
Type 3: $K_4(1,4,4,d',e',f')$	From Lemma 5
Type 4: $K_4(2,3,4,d',e',f')$	From Lemma 4
Type 5: $K_4(2,2,5,d',e',f')$	No solution
Type 6: $K_4(1,2,c',2,e',4)$	No solution
Type 7: $K_4(1,2,c',4,e',2)$	No solution
Type 8: $K_4(1,2,c',3,e',3)$	No solution
Type 9: $K_4(1,3,c',2,e',3)$	No solution
Type 10: $K_4(2,2,c',2,e',3)$	No solution

Similarly to Theorem 13, we can easily prove the following results.

THEOREM 14. K_4-homeomorphs $K_4(1,2,c,3,e,3)$ with girth 9 is χ-unique where $c \geq 6$ and $e \geq 5$.

THEOREM 15. K_4-homeomorphs $K_4(1,3,c,2,e,3)$ with girth 9 is χ-unique for all $c \geq 5$ and $e \geq 5$.

The following results were obtained in [6, 7, 8].

THEOREM 16. K_4-homeomorphs $K_4(1,4,4,d,e,f)$ with girth 9 is not χ-unique if and only if G is isomorphic to

$$\begin{array}{lll} K_4(1,4,4,4,2,6), & K_4(1,4,4,6,2,6), & K_4(1,4,4,2,3,7), \\ K_4(1,4,4,6,3,7), & K_4(1,4,4,6,3,8), & K_4(1,4,4,3,5,8), \\ K_4(1,4,4,i,i+1,i+5), & K_4(1,4,4,i+2,i,i+4), & \end{array}$$

where $i \geq 3$.

THEOREM 17. Let K_4-homeomorphs $K_4(2,3,4,d,e,f)$ with girth 9 is not χ-unique if and only if G is isomorphic to

$$\begin{array}{lll} K_4(2,3,4,1,5,8), & K_4(2,3,4,2,4,8), & K_4(2,3,4,2,6,8), \\ K_4(2,3,4,e+4,e,1), & K_4(2,3,4,6,e,1), & K_4(2,3,4,1,7,f), \end{array}$$

where $e \geq 6$ and $f \geq 4$.

THEOREM 18. K_4-homeomorphs $K_4(1,2,6,d,e,f)$ with girth 9 is not χ-unique if and only if it is isomorphic to

$$\begin{array}{lll} K_4(1,2,6,6,3,4), & K_4(1,2,6,9,3,5), & K_4(1,2,6,5,5,5), \\ K_4(1,2,6,4,5,8), & K_4(1,2,6,3,4,10), & K_4(1,2,6,5,3,8), \\ K_4(1,2,6,4,s,4), & K_4(1,2,6,f+2,4,f), & K_4(1,2,6,i,i+7,i+1), \\ K_4(1,2,6,i+2,i,i+6), & K_4(1,2,6,i,i+1,i+3), & K_4(1,2,6,i+2,i+2,i), \end{array}$$

where $i \geq 1$, $s \geq 4$, $f \geq 4$.

Now, we present the necessary and sufficient conditions for all families of K_4-homeomorphs graph with girth 9 to be χ-unique.

THEOREM 19. Let G be a K_4-homeomorphs graph with girth 9. Then G is not χ-unique if and only if G is isomorphic to

$$\begin{array}{lllll} K_4(2,3,4,1,5,8), & \text{or} & K_4(2,3,4,2,4,8), & \text{or} & K_4(2,3,4,2,6,8), \\ K_4(1,4,4,4,2,6), & \text{or} & K_4(1,4,4,2,3,7), & \text{or} & K_4(1,4,4,6,2,6), \\ K_4(1,4,4,6,3,7), & \text{or} & K_4(1,4,4,6,3,8), & \text{or} & K_4(1,4,4,3,5,8), \\ K_4(1,2,6,9,3,5), & \text{or} & K_4(1,2,6,5,5,5), & \text{or} & K_4(1,2,6,4,5,8), \\ K_4(1,2,6,5,3,8), & \text{or} & K_4(1,3,5,2,8,3), & \text{or} & K_4(1,3,5,4,9,4), \\ K_4(1,3,5,5,7,4), & \text{or} & K_4(1,3,5,7,2,7), & \text{or} & K_4(1,3,5,7,8,2), \\ K_4(1,3,5,i,i+6,i+1), & \text{or} & K_4(1,3,5,i,i+1,i+4), & \text{or} & K_4(1,3,5,i+2,i,i+5), \\ K_4(1,3,5,i+2,i+3,i), & \text{or} & K_4(1,2,6,i,i+7,i+1), & \text{or} & K_4(1,2,6,i+2,i,i+6), \\ K_4(1,2,6,i,i+1,i+3), & \text{or} & K_4(1,2,6,i+2,i+2,i), & \text{or} & K_4(2,3,4,e+4,e,1), \\ K_4(2,3,4,6,e,1), & \text{or} & K_4(1,3,5,5,e,2), & \text{or} & K_4(1,3,5,e+3,2,e), \\ K_4(2,3,4,1,7,f), & \text{or} & K_4(1,2,6,4,f,4), & \text{or} & K_4(1,2,6,f+2,4,f), \\ K_4(1,3,5,2,f,f+4), & \text{or} & K_4(1,4,4,s,s+1,s+5), & \text{or} & K_4(1,4,4,s+2,s,s+4), \end{array}$$

where $i \geq 1$, $e \geq 6$, $f \geq 4$ and $s \geq 3$.

PROOF. The result follows directly from Theorems 13–18.

Conclusion. In this paper, we have completely determined the chromaticity of all families of K_4-homeomorphs with girth 9. The problem on chromaticity of such graphs with girth equal and more than 10 still remains open. Another problem to consider is to investigate the chromaticity of K_4-homeomorphs with exactly two paths of length greater than s, $s \geq 3$.

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