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Pairwise Nearly Compact and Pairwise Nearly Paracompact Spaces and it Application

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Abstract. In this paper, we shall introduce and study the pairwise nearly compact and paracompact bitopological spaces and investigate some of their characterizations. Moreover we study the pairwise nearly compact and pairwise nearly paracompact bitopological spaces by using (i,j)-property. Some examples and counterexamples would be provide in order to establish their properties. Furthermore the Lindelöf and pairwise nearly Lindelöf bitopological spaces are also discussed. Finally, we suggest some ideas to apply the spaces mention above to study the topological entropy which is using to measure the complexity of dynamical systems.

INTRODUCTION

Compactness plays an important role in mathematical analysis especially in the field of topology. Currently, it is still an active research including it generalizations, and to find it applications in other fields. In literature there are several generalizations of the notion of compact and paracompact spaces and these are studied separately for different reasons and purposes. One of the main generalization is Lindelöf spaces. In early 1970's, Singal and Mathur [1, 2] and Herrington [3] have introduced and studied the notion of nearly compact spaces as one of the generalizations of compactness. Thereafter Ergun [4] introduced the notion of nearly paracompact space and studies some of it characterizations.

Next in 1982, Balasubramaniam [5] introduced and studied the notion of nearly Lindelöf spaces as another extension of compactness. Moreover the author and his colleague [6, 7, 8] have been studied the generalizations of compactness and Lindelöfness in bitopological spaces and investigated several of their characterizations. The purpose of this paper is to investigate some properties of generalized compactness in topological spaces and bitopological spaces, and studied some of their properties and characterizations. Section 3 until section 6 will discuss these kinds of purposes.

There are several definitions of topological entropy in literature since 1965 till now, proposed such as by Adler et al. [9], Bowen [10], Cánovas and Rodríguez [11], Yang and Bai [12], etc. In section 7, we review some concepts and properties of topological entropy in the sense of Adler et al. [9]. Adler et al. used the concept of compact space to define topological entropy. As we know, topological entropy have applications in dynamical systems to measure how complicated the systems is or has chaotic property. At the end of this paper we suggests some ideas of topological entropy by using the generalized compact spaces such as nearly compact, paracompact, nearly paracompact, Lindelöf, etc. It might be useful for future research.

PRELIMINARIES

Throughout this paper, all spaces (X,τ) and (X,τ_1,τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively unless explicitly stated. If $\mathcal{P}$ is a topological property, then (τ_i,τ_j)-$\mathcal{P}$ denotes an analogue of this property for τ_i has property $\mathcal{P}$ with respect to τ_j. As we shall see below, sometimes (τ_1,τ_2)-$\mathcal{P} \Leftrightarrow (\tau_2,\tau_1)$-$\mathcal{P}$ (and thus $\Leftrightarrow$ pairwise-$\mathcal{P}$). Also sometimes τ_1-$\mathcal{P} \Leftrightarrow \tau_2$-$\mathcal{P}$ and thus $\mathcal{P} \Leftrightarrow \tau_1$-$\mathcal{P} \wedge \tau_2$-$\mathcal{P}$, i.e., (X,τ_i)

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has property $\mathcal{P}$ for each $i = 1, 2$. Also note that (X, τ_i) has a property $\mathcal{P} \Leftrightarrow (X, \tau_1, \tau_2)$ has a property τ_i-$\mathcal{P}$. The prefixes (τ_i, τ_j)- or τ_i- will be replaced by (i, j)- or i-, respectively, if there is no chance for confusion.

By i-int(A) and i-cl(A), we shall mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the (i, j)-regular open cover of X, etc. If $S \subseteq A \subseteq X$, then $i\text{-int}_A(S)$ and $i\text{-cl}_A(S)$ will be used to denote the interior and closure of S in the subspace A with respect to topology $\tau_i|_A$, respectively. In this paper always $i, j \in \{1, 2\}$ and $i \neq j$. The reader may consult [13] for the detail notations.

Definition 1. *In a topological space* (X, τ), *a set* A *is called regular open if* $A = \text{int}(\text{cl}(A))$ *and regular closed if* $A = \text{cl}(\text{int}(A))$ [14, p. 92].

Definition 2. *Let* (X, τ_1, τ_2) *be a bitopological space. A subset F of X is said to be*
(*i*) *i-open if F is open with respect to* τ_i *in X, F is called open in X if it is both* 1*-open and* 2*-open in X, or equivalently,* $F \in (\tau_1 \cap \tau_2)$ *in* X;
(*ii*) *i-closed if F is closed with respect* τ_i *in X, F is called closed in X if it is both* 1*-closed and* 2*-closed in X, or equivalently,* $X \setminus F \in (\tau_1 \cap \tau_2)$ *in* X;
(*iii*) (i, j)*-regular open* [12] *if* $F = i - \text{int}(j - \text{cl}(F))$, *F is called pairwise regular open if it is both* $(1, 2)$*-regular open and* $(2, 1)$*-regular open*;
(*iv*) (i, j)*-regular closed* [12] *if* $F = i - \text{cl}(j - \text{int}(F))$, *F is called pairwise regular closed if it is both* $(1, 2)$*-regular closed and* $(2, 1)$*-regular closed*;
(*v*) i-clopen if F is both i-closed and i-open set in X, F is called clopen in X if it is both 1-clopen and 2-clopen in X;
(*vi*) (i, j)*-clopen if F is i-closed and j-open set in X, F is called clopen if it is both* $(1, 2)$*-clopen and* $(2, 1)$*-clopen in* X;
where $i, j \in \{1, 2\}$.

Note that in any bitopological space (X, τ_1, τ_2), $\emptyset$ and X are always (i, j)-regular open and (i, j)-regular closed sets. The complement of an (i, j)-regular open set is (i, j)-regular closed and vice versa. If $S \subseteq X$, then i-int(j-cl(S)) is (i, j)-regular open set in X and i-cl(j-int(S)) is (i, j)-regular closed set in X. We also note that arbitrary union of i-open sets in X is i-open, arbitrary intersection of i-closed sets in X is i-closed, finite intersection of i-open sets in X is i-open, finite union of i-closed sets in X is i-closed. But this is not true, in general, for (i, j)-regular open and (i, j)-regular closed sets. In fact the intersection, not necessarily the union, of two (i, j)-regular open sets in X is (i, j)-regular open. The union, not necessarily the intersection, of two (i, j)-regular closed sets in X is (i, j)-regular closed. More general, finite union of (i, j)-regular closed sets in X is again (i, j)-regular closed and finite intersection of (i, j)-regular open sets in X is again (i, j)-regular open.

Note that in $(\mathbb{R}, \tau_u, \tau_s)$ where τ_u is usual topology and τ_s is lower limit topology or Sorgenfrey topology, i.e., topology generated by right half-open intervals (see [15, p. 75]), the union of (τ_u, τ_s)-regular open sets $(1, 5)$ and $(5, 10)$ is $(1, 5) \cup (5, 10)$, which is not (τ_u, τ_s)-regular open set in $\mathbb{R}$ since

$$\tau_u - \text{int}\left(\tau_s - \text{cl}\big((1,5) \cup (5,10)\big)\right) = \tau_u - \text{int}([1, 10)) = (1, 10) \neq (1, 5) \cup (5, 10).$$

The intersection of (τ_u, τ_s)-regular closed sets [1,5] and [5,10] is {5} which is not (τ_u, τ_s)-regular closed set in $\mathbb{R}$ since $\tau_u - \text{cl}(\tau_s - \text{int}(\{5\})) = \emptyset$.

Subspaces of a topological space are one of the significant research in the study of topological spaces. A subset of a topological space inherits a topology of its own, in an obvious way. If (X, τ) is a topological space and $A \subseteq X$, the collection $\tau|_A = \{G \cap A : G \in \tau\}$ is a topology for A, called relative topology or induced topology for A. Any time a topology is used on a subset of a topological space without explicitly being described, it is assumed to be the relative topology.

Definition 3. [16] *A bitopological space X is said to be* (i, j)*-almost regular if for each* $x \in X$ *and for each* (i, j)*-regular open set* V *of* X *containing* x, *there is an* (i, j)*-regular open set* U *such that* $x \in U \subseteq j\text{-cl}(U) \subseteq V$. *X is called pairwise almost regular if it is both* $(1, 2)$*-almost regular and* $(2, 1)$*-almost regular.*

Definition 4. [17, 6, 7] *A bitopological space* (X, τ_1, τ_2) *is said to be i-Lindelöf if the topological space* (X, τ_i) *is Lindelöf. X is called Lindelöf if it is both* 1*-Lindelöf and* 2*-Lindelöf. Equivalently,* (X, τ_1, τ_2) *is Lindelöf if every i-open cover of X has a countable subcover for each* $i = 1, 2$.

Definition 5. [8] *A bitopological space X is said to be* (i,j)*-nearly Lindelöf if for every* i*-open cover* $\{U_\alpha : \alpha \in \Delta\}$ *of X, there exists a countable subset* $\{\alpha_n : n \in \mathbb{N}\}$ *of* Δ *such that* $X = \bigcup_{n\in\mathbb{N}} i - \mathrm{int}\left(j - \mathrm{cl}\left(U_{\alpha_n}\right)\right)$. *X is called pairwise nearly Lindelöf if it is both* $(1,2)$*-nearly Lindelöf and* $(2,1)$*-nearly Lindelöf.*

COMPACT AND PARACOMPACT SPACES

In advanced analysis and topology, the notion of a compact set is of enormous importance. The definition of compactness uses the notion of an open cover, which we now define.

Definition 6. [18] *A topological space X is compact if each open cover of X has a finite subcover.*

Example 1. $\mathbb{R}$ with usual topology is not compact. In fact, the cover of $\mathbb{R}$ by the open sets $(-n,n)$ for $n \in \mathbb{N}$ has no finite subcover.

Paracompact spaces were first introduced by Dieudonné (see [18]) in 1944 as a natural generalization of compact spaces but still retaining enough structure to enjoy many of the properties of compact spaces, yet sufficiently general to include a much wider class of spaces. To proceed, we need a great deal of terminology applying to coverings.

Definition 7. [19, 20] *Let X be a topological space. A cover* $\mathcal{V} = \{V_j : j \in \Gamma\}$ *of X is a refinement of another cover* $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ *if for each* $j \in \Gamma$*, there exists an* $\alpha(j) \in \Delta$ *such that* $V_j \subseteq U_{\alpha(j)}$*, i.e., each* $V \in \mathcal{V}$ *is contained in some* $U \in \mathcal{U}$*. If the elements of* $\mathcal{V}$ *are open sets, we will call* $\mathcal{V}$ *an open refinement of* $\mathcal{U}$; *if they are closed sets, we call* $\mathcal{V}$ *a closed refinement. If* $\mathcal{V}$ *is finite subcollection, we will call* $\mathcal{V}$ *a finite refinement of* $\mathcal{U}$.

In other words, a cover $\mathcal{V}$ is said to be a refinement of a cover $\mathcal{U}$, denoted as $\mathcal{U} \prec \mathcal{V}$, if every member of $\mathcal{V}$ is a subset of some members of $\mathcal{U}$. We will use this notation in the investigation of topological entropy.

Definition 8. [19, 20] *A family* $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ *of subsets of a space* (X,τ) *is locally finite if for every point* $x \in X$*, there exists a neighbourhood* N_x *of* x *such that the set* $\{\alpha \in \Delta : N_x \cap U_\alpha \neq \emptyset\}$ *is finite, i.e., each* $x \in X$ *has a neighbourhood* N_x *meeting only finitely many* $U \in \mathcal{U}$.

To see how paracompactness generalizes compactness, first we write an equivalent way of the definition of compactnes as follows:

"*A topological space X is compact if each open cover of X has an open finite refinement*."

This definition is equivalent to the usual one; given a refinement $\mathcal{V}$, one can choose for each element of $\mathcal{V}$ an element of $\mathcal{U}$ containing it; in this way one obtains a finite subcollection of $\mathcal{U}$ that covers *X*.

Definition 9. [18, 20] *A topological space X is paracompact if each open cover of X has an open locally finite refinement.*

Many authors, folowing the lead of Bourbaki [21], include as part of the definition of the term paracompact the requirement that the space is Hausdorff. Bourbaki also includes the Hausdorff condition as part of the definition of compact. We shall not follow this convention.

NEARLY COMPACT AND NEARLY PARACOMPACT SPACES

A topological space (X,τ) is said to be nearly compact [1] if and only if every open cover of X has a finite subcollection, the interiors of the closures of whose members cover the space X. A subset of a topological space is called nearly compact if it is a nearly compact space as a subspace. Now by using Definition 1, we are ready to write another equivalent definition of a nearly compact space. A topological space (X,τ) is nearly compact if and only if each regular open cover of X has a finite subcover [1]. The following is another equivalent definition of nearly compact space.

Definition 10. *A topological space* (X,τ) *is said to be nearly compact if for every open cover* $\{U_\alpha : \alpha \in \Delta\}$ *of X, there exists a finite subset* $\{\alpha_k : k = 1, \dots, n\}$ *of* Δ *such that* $X = \bigcup_{k=1}^{n} \mathrm{int}\left(\mathrm{cl}\left(U_{\alpha_k}\right)\right)$.

In [22], Singal and Arya have introduced a class of topological spaces called nearly paracompact spaces, see also [4, 23]. These are characterized by the following property:

Definition 11. *A topological space X is said to be nearly paracompact if every regular open cover X admits an open locally finite refinement. A subset of X is called nearly paracompact if the relative topology defined on it is nearly paracompact.*
Note that, if A is an open set, then for any subset B

$$A \cap \operatorname{int}\big(\operatorname{cl}(B)\big) \subseteq \operatorname{int}\big(\operatorname{cl}(A \cap B)\big). \tag{1}$$

Theorem 1. *A topological space is nearly paracompact if and only if every regular open cover admits a regular open locally finite refinement.*
Proof. Necessity: Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a regular open cover of any nearly paracompact space X. Then $\mathcal{U}$ has an open locally finite refinement, say, $\mathcal{V} = \{V_j : j \in \Gamma\}$. Hence for every $x \in X$, there exists an open neighbourhood N_x of x such that the set $\{j \in \Gamma : N_x \cap V_j \neq \emptyset\}$ is finite. On the other hand $\mathcal{V}^* = \left\{\operatorname{int}\left(\operatorname{cl}(V_j)\right) : V_j \in \mathcal{V}\right\}$ is the family of regular open subsets of X envelopes of members of $\mathcal{V}$ trivially a cover of X. Since $N_x \cap \operatorname{int}\left(\operatorname{cl}(V_j)\right) \subseteq \operatorname{int}\left(\operatorname{cl}(N_x \cap V_j)\right)$ by (1), then $\left\{j \in \Gamma : N_x \cap \operatorname{int}\left(\operatorname{cl}(V_j)\right) \neq \emptyset\right\}$ is finite for every $x \in X$ and thus $\mathcal{V}^*$ is locally finite. It is also a refinement of $\mathcal{U}$ since for any $V_j \in \mathcal{V}$, there exists a superset $U_{\alpha(j)} \in \mathcal{U}$ such that $V_j \subseteq U_{\alpha(j)}$ and consequently $\operatorname{int}\left(\operatorname{cl}(V_j)\right) \subseteq U_{\alpha(j)}$ holds. Sufficiency is straightforward. ■

Every paracompact space is evidently nearly paracompact but the converse is not hold in general as shown by [4]. We summarize it as the following counterexample.

Example 2. Let $\mathbb{R}$ be a set of all real numbers and let τ denote a topology on $\mathbb{R}$ such that the whole proper subsets are dense. Then $\mathbb{R}$ determined by τ is nearly paracompact since the only nonempty regular open subset is $\mathbb{R}$. But it is not paracompact since it is impossible to define an open locally finite refinement of the cover with basic neighborhoods. For example, we can choose the basic neighborhoods $G_x(\varepsilon) = (x - \varepsilon,\ \infty)$ on $\mathbb{R}$ for any $x \in \mathbb{R}$ and $\varepsilon > 0$. This fit the above description.

PAIRWISE NEARLY COMPACT SPACES

In this section, the definitions of pairwise nearly compact spaces will be introduced and some characterizations are established.

Definition 12. *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-nearly compact if for every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset $\{\alpha_k : k = 1, \ldots, n\}$ of Δ such that $X = \bigcup_{k=1}^{n} i - \operatorname{int}\left(j - \operatorname{cl}(U_{\alpha_k})\right)$, or equivalently, every (i,j)-regular open cover of X has a finite subcover. X is called pairwise nearly compact if it is both $(1,2)$-nearly compact and $(2,1)$-nearly compact.*

Example 3. The space $(\mathbb{R}, \tau_u, \tau_s)$ where τ_u is usual topology and τ_s is lower limit topology, is not (τ_u, τ_s)-nearly compact since the (τ_u, τ_s)-regular open cover $\{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ has no finite subcover. It is also not (τ_s, τ_u)-nearly compact since the (τ_s, τ_u)-regular open cover $\{[-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ has no finite subcover. However, the spaces $(\mathbb{R}, \tau_u, \tau_{cof})$, $(\mathbb{R}, \tau_u, \tau_{coc})$ and $(\mathbb{R}, \tau_{cof}, \tau_{coc})$ are (τ_u, τ_{cof})-nearly compact, (τ_u, τ_{coc})-nearly compact and (τ_{cof}, τ_{coc})-nearly compact, respectively since the only (τ_u, τ_{cof})-regular open cover, (τ_u, τ_{coc})-regular open cover and (τ_{cof}, τ_{coc})-regular open cover of these spaces is $\{\mathbb{R}\}$ which has one element, so it is finite. Here, τ_{cof}, and τ_{coc} mean cofinite topology and cocountable topology, respectively.

We easily note that every i-compact space is (i,j)-nearly compact but the converse is not true. For example, the space $(\mathbb{R}, \tau_{l.r}, \tau_{cof})$ is $(\tau_{l.r}, \tau_{cof})$-nearly compact since the $(\tau_{l.r}, \tau_{cof})$-regular open cover of $\mathbb{R}$ is $\{\mathbb{R}\}$ which has one element, so it is finite. Recall that τ_{cof} is cofinite topology and $\tau_{l.r}$ is left ray topology on $\mathbb{R}$ generated by basis sets of the form $\{(-\infty, x) : x \in \mathbb{R}\}$. But $(\mathbb{R}, \tau_{l.r}, \tau_{cof})$ is not $\tau_{l.r}$-compact since $\{(-\infty, n) : n \in \mathbb{N}\}$ a $\tau_{l.r}$-open cover of $\mathbb{R}$ has no finite subcover.

It is well known that the i-closed subspaces of an i-compact space are i-compact but for (i,j)-nearly compact spaces it is not necessarily true in general by Example 4 below shows. The following proposition gives an analogue result for (i,j)-nearly compact subspaces.

Proposition 1. *Let (X, τ_1, τ_2) be an (i,j)-nearly compact space and let A be an (i,j)-clopen and i-open subspace of X. Then $(A, \tau_1|_A, \tau_2|_A)$ is (i,j)-nearly compact.*
Proof. Let A be an (i,j)-clopen and i-open subspace of X. Let $\{U_\alpha : \alpha \in \Delta\}$ be an i-open cover of A. Since i-open subsets of an i-open subspace of X is i-open sets in X, then $\{U_\alpha : \alpha \in \Delta\}$ is a cover of A by i-open subsets of X. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ forms an i-open cover of X. Since X is (i,j)-nearly compact, there exists a finite subfamily $\{X \setminus A, U_{\alpha_1}, \dots, U_{\alpha_n}\}$ such that

$$X = \left(\bigcup_{k=1}^{n} i - \mathrm{int}_X \left(j - \mathrm{cl}_X(U_{\alpha_k})\right)\right) \cup \left(i - \mathrm{int}_X\left(j - \mathrm{cl}_X(X \setminus A)\right)\right)$$
$$= \left(\cup_{k=1}^{n} i - \mathrm{int}_X \left(j - \mathrm{cl}_X(U_{\alpha_k})\right)\right) \cup (X \setminus A).$$

But A and $X \setminus A$ are disjoint; hence $A \subseteq \cup_{k=1}^{n} i - \mathrm{int}_X \left(j - \mathrm{cl}_X(U_{\alpha_k})\right)$. Thus

$$A = A \cap \left(\bigcup_{k=1}^{n} i - \mathrm{int}_X \left(j - \mathrm{cl}_X(U_{\alpha_k})\right)\right)$$
$$= \bigcup_{k=1}^{n} \left(A \cap i - \mathrm{int}_X \left(j - \mathrm{cl}_X(U_{\alpha_k})\right)\right)$$
$$= \bigcup_{k=1}^{n} i - \mathrm{int}_A \left(A \cap j - \mathrm{cl}_X(U_{\alpha_k})\right)$$
$$= \cup_{k=1}^{n} i - \mathrm{int}_A \left(j - \mathrm{cl}_A(U_{\alpha_k})\right).$$

Therefore $(A, \tau_1|_A, \tau_2|_A)$ is (i,j)-nearly compact. ■

Corollary 1. *Let (X, τ_1, τ_2) be a pairwise nearly compact space and let A be a clopen subspace of X. Then $(A, \tau_1|_A, \tau_2|_A)$ is pairwise nearly compact.*

In Proposition 1, the condition that A is (i,j)-clopen and i-open subspace is necessary. It is not sufficient to assume that A is only i-closed as the following example shows.

Example 4. The space $(\mathbb{R}, \tau_{coc}, \tau_{cof})$ is clearly (τ_{coc}, τ_{cof})-nearly compact since the only nonempty (τ_{coc}, τ_{cof})-regular open subset is $\mathbb{R}$. Let $\mathbb{N}$ be the set of all natural numbers, then $\mathbb{N}$ is τ_{coc}-closed but not τ_{coc}-open. The subspace $(\mathbb{N}, \tau_{coc}|_{\mathbb{N}}, \tau_{cof}|_{\mathbb{N}})$ of $\mathbb{R}$ where $\tau_{coc}|_{\mathbb{N}}$ is discrete topology on $\mathbb{N}$ is not $(\tau_{coc}|_{\mathbb{N}}, \tau_{cof}|_{\mathbb{N}})$-nearly compact since $\{\{x\} : x \in \mathbb{N}\}$ is $(\tau_{coc}|_{\mathbb{N}}, \tau_{cof}|_{\mathbb{N}})$-regular open cover of $\mathbb{N}$, has no finite subcover.

PAIRWISE NEARLY PARACOMPACT SPACES

We have already defined the concept of locally finite for a topological space (X, τ) in Definition 8. If the bitopological space (X, τ_1, τ_2) considered, i-locally finite concept appear as in the following definition.

Definition 13. *A family $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ of subsets of a bitopological space (X, τ_1, τ_2) is i-locally finite if for every point $x \in X$, there exists an i-neighbourhood U_x of x such that the set $\{\alpha \in \Delta : U_x \cap U_\alpha \neq \emptyset\}$ is finite, i.e., each $x \in X$ has an i-neighbourhood U_x meeting only finitely many $U \in \mathcal{U}$.*
We extend the notion of paracompactness to the bitopological setting as follows.

Definition 14. *A bitopological space X is said to be i-paracompact if every i-open cover of X admits an i-open refinement which is i-locally finite. X is called paracompact if it is i-paracompact for each $i = 1, 2$.*

In 1969, Singal and Arya [22] introduced the notion of nearly paracompact in topological spaces. We extend this notion to bitopological spaces as follows.

Definition 15. *A bitopological space X is said to be (i,j)-nearly paracompact if every cover of X by (i,j)-regular open sets admits an i-open refinement which is i-locally finite. X is called pairwise nearly paracompact if it is both $(1,2)$-nearly paracompact and $(2,1)$-nearly paracompact.*
It is clear that every i-compact space is i-paracompact but the converse is not true as the following example shows.

Example 5. The space $(\mathbb{R}, \tau_u, \tau_s)$ is τ_u-paracompact but it is not τ_u-compact (see [15, p. 56-57]). It is also not (τ_u, τ_s)-nearly compact by Example 3.

Proposition 2. [8] *Let (X, τ_1, τ_2) be an (i, j)-almost regular and (i, j)-nearly Lindelöf space. Then X is (i, j)-nearly paracompact.*

Clearly, every i-paracompact space is (i, j)-nearly paracompact but the converse is not true as Example 6 below shows. Furthermore, every (i, j)-nearly compact space is (i, j)-nearly paracompact but the converse is not true as Example 7 below shows.

Example 6. Observe that the spase $(\mathbb{R}, \tau_{coc}, \tau_{cof})$ is a (τ_{coc}, τ_{cof})-nearly compact and hence (τ_{coc}, τ_{cof})-nearly Lindelöf. Thus $\mathbb{R}$ is a (τ_{coc}, τ_{cof})-nearly paracompact by Proposition 2, since $\mathbb{R}$ is (τ_{coc}, τ_{cof})-almost regular. But $\mathbb{R}$ is not τ_{coc}-paracompact since the intersection of any countable collection of τ_{coc}-open sets is τ_{coc}-open and thus uncountable (see [15, p. 51]).

Example 7. The space $(\mathbb{R}, \tau_u, \tau_s)$ is not (τ_u, τ_s)-nearly compact by Example 3 but $\mathbb{R}$ is (τ_u, τ_s)-nearly paracompact by Proposition 2, since it is (τ_u, τ_s)-almost regular and (τ_u, τ_s)-nearly Lindelöf.

Observe that, i-paracompact and (i, j)-nearly compact spaces are independent notions as Example 5 and Example 6 above show. Now the following diagram holds.

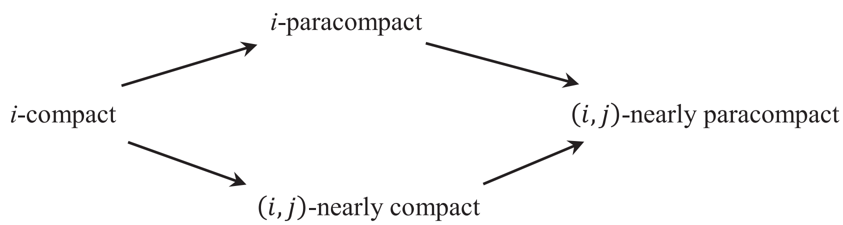

FIGURE 1. Relations among generalized i-compact spaces.

None of these implications is reversible by the necessary counterexamples above. The diagram for pairwise properties is similar. Many of the concepts and results obtained will be very useful for the future research.

TOPOLOGICAL ENTROPY OF MAPS ON COMPACT SPACES

Topological entropy can be an indicator of complicated (chaotic) behavior in dynamical systems. Whether the topological entropy of a dynamical system is positive or not is of primary significance, due to the fact that positive topological entropy implies that one can assert that the system is chaotic. It is hard, as remarked by [24, 12], to get a good idea of what topological entropy means directly from various definitions of topological entropy. Thus it is enough to know that topological entropy of a dynamical system is a measure of complexity of dynamic behavior of the system, and it can be seen as a quantitative measurement of how chaotic of a dynamical system. Generally speaking, the larger the topological entropy of a system is, the more complicated the dynamics of this system would be.

The characterization of chaos in terms of topological entropy is the most satisfactory one from a mathematical perspective but is not very computable in applications (with a computer). The definition in terms of Lyapunov exponents is the most computable (possible to estimate) on a computer.

Most of the results and proofs of this section are taken from Adler et al. [9]. In this section, we always let X be a nonempty compact space unless explicitly stated.

Definition 16. *For any open cover $\mathcal{U}$ of X, let $N(\mathcal{U})$ denote the number of sets in a subcover of minimal cardinality. A subcover of a cover is minimal if no other subcover contains fewer members. We call $H(\mathcal{U}) = \log N(\mathcal{U})$ the topological entropy of $\mathcal{U}$.*

Since X is compact and $\mathcal{U}$ is an open cover, there always exists a finite subcover. Then we have

$$H(\mathcal{U}) = \log \min\{\text{card}(\mathcal{V}) : \mathcal{V} \text{ a finite subcover of } \mathcal{U}\}.$$

Definition 17. *For any two covers $\mathcal{U}, \mathcal{V}$ of X, $\mathcal{U} \vee \mathcal{V} \equiv \{A \cap B : A \in \mathcal{U}, B \in \mathcal{V}\}$ defines their join.*

Let $f: X \to X$ be a continuous mapping. If $\mathcal{U}$ is an open cover of X then from continuity, the family $f^{-1}(\mathcal{U}) = \{f^{-1}(A) : A \in \mathcal{U}\}$ is again an open cover of X.

Remark 1. (*i*) *The operation* $\vee$ *is obviously commutative and associative.*
(*ii*) $f^{-1}(\vee_{i=0}^{n-1}\mathcal{U}_i)=\vee_{i=0}^{n-1}f^{-1}(\mathcal{U}_i)$.

The following proposition state the properties of $N(\mathcal{U})$ and $H(\mathcal{U})$. The reader may find the detail proof in [25].

Proposition 3. *Let* $\mathcal{U},\mathcal{V}$ *be two open covers of a compact space* X. *Then*
(*i*) $N(\mathcal{U}\vee\mathcal{V})\le N(\mathcal{U})\cdot N(\mathcal{V})$ *and* $H(\mathcal{U}\vee\mathcal{V})\le H(\mathcal{U})+H(\mathcal{V})$;
(*ii*) $\mathcal{U}\prec\mathcal{V}\Longrightarrow f^{-1}(\mathcal{U})\prec f^{-1}(\mathcal{V})$;
(*iii*) $f^{-1}(\mathcal{U}\vee\mathcal{V})=f^{-1}(\mathcal{U})\vee f^{-1}(\mathcal{V})$;
(*iv*) $N(\mathcal{U})\ge N\big(f^{-1}(\mathcal{U})\big)$ *and* $H(\mathcal{U})\ge H\big(f^{-1}(\mathcal{U})\big)$. *If* f *is onto, we have equality.*

Lemma 1. *For every open cover* $\mathcal{U}$ *of* X, *the following*

$$\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{k=0}^{n-1}f^{-k}(\mathcal{U})\right)=\lim_{n\to\infty}\frac{1}{n}H\big(\mathcal{U}\vee f^{-1}(\mathcal{U})\vee\cdots\vee f^{-n+1}(\mathcal{U})\big)$$

exists and is a nonnegative real number.
Proof. See [9, 25].

Definition 18. *The topological entropy* $h(f,\mathcal{U})$ *of a mapping* f *with respect to an open cover* $\mathcal{U}$ *is defined as*

$$h(f,\mathcal{U}):=\lim_{n\to\infty}\tfrac{1}{n}H(\vee_{k=0}^{n-1}f^{-k}(\mathcal{U})).$$

Definition 19. *The topological entropy* $h(f)$ *of a mapping* f *is defined as*

$$h(f):=\sup\{h(f,\mathcal{U})\colon \mathcal{U}\ is\ an\ open\ cover\ of\ X\}.$$

Obviously, $h(f)\in[0,\infty]$.

With several notions of generalized compactness that we have discussed above, we suggest some ideas of new topological entropy by using the generalized compact spaces such as nearly compact, paracompact, nearly paracompact, Lindelöf, etc, and it properties can be investigated. We also can try to investigate topological property in the case of bitopological spaces since there is no researcher doing this nowaday. Many open problems might be appear from this ideas and some problems in dynamical systems can be solved. It might be useful for future research.

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