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A Differential Transformation Method for Approximating the Chaotic Chen System

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Abstract

This paper presents approximate analytical solutions for chaotic Chen system which is a three-dimensional system of ordinary differential equations using differential transform method. Comparisons between the fourth-order Runge-Kutta (RK4) methods with different time steps were done. It has been observed that the accuracy of RK4 solutions can be increased by decreasing the time step. Furthermore, the numerical results are compared to those obtained by the Runge-Kutta method to illustrate the preciseness and effectiveness of the proposed method.

Keywords: Differential Transform Method, Chaotic Chen System, Runge-Kutta Method

1 Introduction

Scientists who deal with nonlinear dynamical systems cannot elude the experience of chaos, an advanced field of mathematics that involves the study of dynamical systems. Many scientists have struggled to find analytical solutions for these chaotic systems. Such tasks always meet with a stumbling block due to its complexities. The accuracies on its long term solutions are sometimes questionable although the numerical methods are available to provide approximate solutions. To overcome this problem, there is a method called the differential transformation method

(DTM). DTM was introduced by Zhou [15], who solved linear and nonlinear initial value problems in electric circuit analysis. DTM was applied to solve many solution of linear and nonlinear differential equation. For example, Al-Sawalha and Noorani [1] have applied this method to solve Lorenz system. Chen and Liu [3] also applied this method to solve two boundary value problems. Besides that, Yu and Chen [14] apply the differential transformation method to the optimization of the rectangular fins with variable thermal parameters.

Differential transformation method is a semi-analytical solution in the form of a polynomial which is different from the traditionally high-ordered Taylor series method. The Taylor series method will consume long computational time for large orders since it requires symbolic computation of the necessary derivatives of the data functions. The differential transform method is an iterative procedure that is described by the transformed equations of original functions for solution of differential equations. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations.

When dealing with nonlinear systems of ordinary differential equations, such as the Chen system, it is often difficult to obtain a closed form of the analytic solution. In the absence of such a solution, the accuracy of the DTM method is then tested against classical numerical methods, such as the Runge–Kutta method (RK4). RK4 has been widely and commonly used for simulating solutions for chaotic systems [7, 13, 8, 9]. The Chen system can exhibit both chaotic and non-chaotic behavior for distinct parameter values. A similar implementation and analysis was done by Hussain and Salleh [6] by using continuous Galerkin Petrov time discretization scheme for the solutions of the Chen system. As such, the objective of this paper is essentially two fold. First, we shall give a comparison in the case of a fixed time step between the DTM and RK4 for the solution of the chaotic Chen system. Secondly, we look into the effect of time steps on the accuracy of the DTM as the Chen system transform from a non-chaotic system to a chaotic one.

2 Differential Transformation Method

The basic definitions of differential transformation are introduced as follows:
Let $\boldsymbol{x}(\boldsymbol{t})$ be analytic in a domain $\mathbf{D}$ and let $\boldsymbol{t}=\boldsymbol{t_i}$ represent any point in $\mathbf{D}$. The function $\boldsymbol{x}(\boldsymbol{t})$ is then represented by one power series whose center is located at $\boldsymbol{t_i}$. The Taylor series expansion function of $\boldsymbol{x}(\boldsymbol{t})$ is of the form:

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \qquad \forall\ t\in\mathbf{D}. \tag{1}$$

The particular case of Eq. (1) when $t_i=0$ is referred to as the Maclaurin series of $x(t)$ and is expressed as:

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \qquad \forall\ t\in\mathbf{D}. \tag{2}$$

Put

$$\varphi(t,k) = \frac{d^k x(t)}{dt^k} \qquad \forall\, t \in \mathbf{D}. \tag{3}$$

For $t = t_i$ then $\varphi(t,k) = \varphi(t_i,k)$, where k belongs to a set of non-negative integers, denoted as the K domain. Thus, (3) can be written as

$$X(k) = \varphi(t_i,k) = \left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i}, \tag{4}$$

where $X(k)$ is called the spectrum of $x(t)$ at $t = t_i$.

If $x(t)$ can be expressed by Taylor series then $x(t)$ can be represented as

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} X(k). \tag{5}$$

Eq. (5) is called the inverse transformation of $X(k)$. Using the symbol "D" denoting the differential transformation process and combining (4) and (5), it is obtained that

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} X(k) \equiv D^{-1}\{X(k)\}, \tag{6}$$

or

$$D[x(t)] \equiv X(k).$$

From Zhou [15], the differential transform of function $x(t)$ is defined as:

$$X(k) = \frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \quad , \qquad k = 0,1,2,\ldots \tag{7}$$

where $X(k)$ represents the transformed function (T-function) and $x(t)$ is the original function. The differential spectrum of $X(k)$ is confined within the interval $t \in [0,H]$, where H is a constant which is the time horizon of interest. The differential inverse transform of $X(k)$ is defined as follows:

$$x(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H}\right)^k X(k). \tag{8}$$

Eq. (8) can be obtained as follows:

Form Eq. (2), we know that $x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}$. Then expand it, we get

$$x(t) = \frac{t^0}{0!}\left[\frac{d^0 x(t)}{dt^0}\right]_{t=0} + \frac{t^1}{1!}\left[\frac{d^1 x(t)}{dt^1}\right]_{t=0} + \frac{t^2}{2!}\left[\frac{d^2 x(t)}{dt^2}\right]_{t=0} + \frac{t^3}{3!}\left[\frac{d^3 x(t)}{dt^3}\right]_{t=0} + \cdots. \tag{9}$$

From Eq. (7), we get

$$X(0) = \frac{H^0}{0!}\left[\frac{d^0 x(t)}{dt^0}\right]_{t=0},$$

$$X(1) = \frac{H^1}{1!}\left[\frac{d^1 x(t)}{dt^1}\right]_{t=0},$$

$$X(2) = \frac{H^2}{2!}\left[\frac{d^2 x(t)}{dt^2}\right]_{t=0},$$

$$X(3) = \frac{H^3}{3!}\left[\frac{d^3 x(t)}{dt^3}\right]_{t=0},$$

$$\vdots$$

The first term in Eq. (9) is

$$\frac{t^0}{0!}\left[\frac{d^0x(t)}{dt^0}\right]_{t=0} = \frac{t^0}{H^O}\left[\frac{H^0}{0!}\left[\frac{d^0x(t)}{dt^0}\right]_{t=0}\right].$$

Proceeding in a similar manner, we can obtain second term, third term and so on. Therefore, Eq. (9) becomes

$$x(t) = \frac{t^0}{H^O}\left[\frac{H^0}{0!}\left[\frac{d^0x(t)}{dt^0}\right]_{t=0}\right] + \frac{t^1}{H^1}\left[\frac{H^1}{1!}\left[\frac{d^1x(t)}{dt^1}\right]_{t=0}\right] + \frac{t^2}{H^2}\left[\frac{H^2}{2!}\left[\frac{d^2x(t)}{dt^2}\right]_{t=0}\right] + \frac{t^3}{H^3}\left[\frac{H^3}{3!}\left[\frac{d^3x(t)}{dt^3}\right]_{t=0}\right] + \cdots.$$

Because $X(k) = \frac{H^k}{k!}\left[\frac{d^kx(t)}{dt^k}\right]_{t=0}$, then

$$x(t) = \frac{t^0}{H^O}X(0) + \frac{t^1}{H^1}X(1) + \frac{t^2}{H^2}X(2) + \frac{t^3}{H^3}X(3) + \cdots.$$

Hence,

$$x(t) = \sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^k X(k). \quad ■$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The original functions are denoted by lowercase letters, while their transformed functions (i.e., their *T*-functions) are indicated by the corresponding uppercase letter. The values of function $X(k)$ at values of argument k are referred to as discretes, i.e., $X(0)$ is known as the zero discrete, $X(1)$ as the first discrete etc. The more discretes available, the more precise it is possible to restore the unknown function. The function $x(t)$ consists of the *T*-function $X(k)$, and its value is given by the sum of the *T*-function with $\left(\frac{t}{H}\right)^k$ as its coefficient.

Using the differential transformation, a differential equation in the time domain can be transformed to be an algebraic equation in the K domain and $x(t)$ can be obtained by finite-term Taylor series plus a remainder, as Eq. (2)

$$x(t) = \sum_{k=0}^{n}\frac{t^k}{k!}X(k) + R_{n+1}(t), \tag{10}$$

where

$$R_{n+1}(t) = \sum_{k=n+1}^{\infty}\frac{t^k}{k!}X(k),$$

and $R_{n+1}(t) \to 0$, as $n+1 \to \infty$ within the time interval of interest, say $t \in [0, H]$. If $R_{n+1}(t)$ is small enough, then $x(t)$ can be represented by finite terms. Therefore, the function $x(t)$ is expressed by a finite series and Eq. (8) can be written as:

$$x(t) = \sum_{k=0}^{n}\left(\frac{t}{H}\right)^k X(k) \tag{11}$$

Eq. (11) implies that the value of $\sum_{k=n+1}^{\infty}\left(\frac{t}{H}\right)^k X(k)$ is negligible. Usually, the value of n are decided by a convergence of series coefficient.

There are four operation properties of differential transformation have been used in this paper. Let us consider $h(t)$, $f(t)$ and $g(t)$ are three uncorrelated functions with time t and $H(k)$, $F(k)$, $G(k)$ are the transformed functions corresponding to $h(t)$, $f(t)$, $g(t)$, respectively. Let $\lambda \in \mathbb{R}$, the following theorem that can be deduced from Eqs. (7) and (8) are given below [8].

Theorem 2.1. If $h(t) = f(t) \pm g(t)$, then $H(k) = F(k) \pm G(k)$.

Proof. Suppose that $h(t) = f(t) \pm g(t)$. Then

$$D[h(t)] = D[f(t)] \pm D[g(t)].$$

Since $D[h(t)] = H(k)$, $D[f(t)] = F(k)$, and $D[g(t)] = G(k)$, hence

$$H(k) = F(k) \pm G(k). \ \blacksquare$$

Theorem 2.2. If $h(t) = \lambda f(t)$, then $H(k) = \lambda F(k)$, where λ is a constant.

Proof. Suppose that $h(t) = \lambda f(t)$. Then

$$D[h(t)] = D[\lambda f(t)] = \lambda D[f(t)].$$

Because $D[h(t)] = H(k)$ and $D[f(t)] = F(k)$, hence

$$H(k) = \lambda F(k). \ \blacksquare$$

Theorem 2.3. If $h(t) = \frac{df(t)}{dt}$, then $H(k) = \frac{k+1}{H} F(k+1)$.

Proof. Suppose that $h(t) = \frac{df(t)}{dt}$. Then

$$D[h(t)] = D[\frac{df(t)}{dt}].$$

We have from Eq. (7),

$$H(k) = \frac{H^k}{k!}\left[\frac{d^k}{dt^k} h(t)\right]_{t=0}.$$

Since $h(t) = \frac{df(t)}{dt}$, then

$$H(k) = \frac{H^k}{k!}\left[\frac{d^k}{dt^k}\frac{df(t)}{dt}\right]_{t=0}.$$

Next, multiply the numerator and denominator by H and $k+1$, we get

$$H(k) = \frac{H^{k+1}(k+1)}{H(k+1)!}\left[\frac{d^{k+1}f(t)}{dt^{k+1}}\right]_{t=0} = \frac{k+1}{H}\left[\frac{H^{k+1}}{(k+1)!}\left[\frac{d^{k+1}f(t)}{dt^{k+1}}\right]_{t=0}\right].$$

From Eq. (7), we know that

$$F(k+1) = \frac{H^{k+1}}{(k+1)!}\left[\frac{d^{k+1}f(t)}{dt^{k+1}}\right]_{t=0}.$$

Hence,

$$H(k) = \frac{k+1}{H} F(k+1). \quad \blacksquare$$

Theorem 2.4. If $h(t) = f(t)g(t)$, then $H(k) = \sum_{l=0}^{k} F(l)G(k-l)$.

Proof. Suppose that $h(t) = f(t)g(t)$. From Eq. (8), we get

$$h(t) = \left[\sum_{k=0}^{\infty} \left(\frac{t}{H}\right)^k F(k)\right]\left[\sum_{k=0}^{\infty} \left(\frac{t}{H}\right)^k G(k)\right].$$

Then,

$$h(t) = \left[F(0) + \frac{t}{H}F(1) + \left(\frac{t}{H}\right)^2 F(2) + \cdots\right]\left[G(0) + \frac{t}{H}G(1) + \left(\frac{t}{H}\right)^2 G(2) + \cdots\right].$$

Then expand it, we get

$$h(t) = F(0)G(0) + \left(\frac{t}{H}\right)[F(0)G(1) + F(1)G(0)] + \left(\frac{t}{H}\right)^2 [F(0)G(2) + F(1)G(1) + F(2)G(0)] + \cdots.$$

Hence,

$$h(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H}\right)^k \sum_{l=0}^{k} F(l)G(k-l).$$

From Eq. (8), we know that

$$h(t) = \sum_{k=0}^{\infty} \left(\frac{t}{H}\right)^k H(k).$$

Therefore,

$$H(k) = \sum_{l=0}^{k} F(l)G(k-l). \quad \blacksquare$$

3 Chen System

The Chen dynamical system, first found by Chen and Ueta [4], is defined as:

$$\frac{dx}{dt} = a(y-x), \tag{12}$$

$$\frac{dy}{dt} = (c-a)x - xz + cy, \tag{13}$$

$$\frac{dz}{dt} = xy - bz. \tag{14}$$

where x, y and z are state variables and parameters a, b and c are positive real numbers. Bifurcation studies shows that with the parameters $a = 35$ and $c = 28$ and $b = 12$, system (12)-(14) exhibits non-chaotic behaviour while for parameters $a = 35$ and $c = 28$ and $b = 3$, the system exhibits chaotic behaviour [11]. For other aspects of this dynamical system, see for example, [13, 5, 6, 11].

By taking the differential transform of Eqs. (12)-(14) with respect to time t, gives:

$$\frac{k+1}{H}X(k+1) = -aX(k) + aY(k), \tag{15}$$

$$\frac{k+1}{H}Y(k+1) = (c-a)X(k) - \sum_{l=0}^{k} X(l)Z(k-l) + cY(k), \tag{16}$$

$$\frac{k+1}{H}Z(k+1) = \sum_{l=0}^{k} X(l)Y(k-l) - bZ(k). \tag{17}$$

where $X(k)$, $Y(k)$ and $Z(k)$ are the differential transformations of the corresponding functions $x(t)$, $y(t)$ and $z(t)$, respectively.

***Proof Eq. (15)*:** From Eq. (12), $\frac{dx}{dt} = ay - ax$, and then

$$D\left[\frac{dx}{dt}\right] = D[ay] - D[ax].$$

By Theorem 3,

$$D\left[\frac{dx}{dt}\right] = \frac{k+1}{H}X(k+1)$$

and by Theorem 2,

$$D[ay] = aD[y] = aY(k),\ D[ax] = aD[x] = aX(k).$$

Hence,

$$\frac{k+1}{H}X(k+1) = -aX(k) + aY(k).\ \blacksquare$$

***Proof Eq. (16)*:** From Eq. (13), $\frac{dy}{dt} = (c-a)x - xz + cy$. Then

$$\frac{dy}{dt} = cx - ax - xz + cy,$$

and then

$$\mathrm{D}\left[\frac{dy}{dt}\right] = D[cx] - D[ax] - D[xz] + D[cy].$$

By Theorem 3,

$$D\left[\frac{dy}{dt}\right] = \frac{k+1}{H}Y(k+1),$$

and by Theorems 2 and 4,

$$D[cx] = cD[x] = cX(k),$$
$$D[ax] = aD[x] = aX(k),$$
$$D[cy] = cD[y] = cY(k),$$
$$D[xz] = \sum_{l=0}^{k} X(l)Z(k-l).$$

Hence,

$$\frac{k+1}{H}Y(k+1) = cX(k) - aX(k) - \sum_{l=0}^{k} X(l)Z(k-l) + cY(k).$$

Thus by factorization, we obtain

$$\frac{k+1}{H}Y(k+1) = (c-a)X(k) - \sum_{l=0}^{k} X(l)Z(k-l) + cY(k). \ \blacksquare$$

***Proof Eq. (17)*:** From Eq. (14), $\frac{dz}{dt} = xy - bz$. Then

$$D\left[\frac{dz}{dt}\right] = D[xy] - D[bz].$$

By Theorem 3,

$$D\left[\frac{dz}{dt}\right] = \frac{k+1}{H}Z(k+1).$$

and by Theorems 4 and 2,

$$D[xy] = \sum_{l=0}^{k} X(l)Y(k-l),$$
$$D[bz] = bD[z] = bZ(k).$$

Hence,

$$\frac{k+1}{H}Z(k+1) = \sum_{l=0}^{k} X(l)Y(k-l) - bZ(k). \ \blacksquare$$

Eqs. (15)-(17) can be rewritten in the following forms:

$$X(k+1) = \frac{H}{k+1}[-aX(k) + aY(k)], \quad (18)$$

$$Y(k+1) = \frac{H}{k+1}\left[(c-a)X(k) - \sum_{l=0}^{k} X(l)Z(k-l) + cY(k)\right], \quad (19)$$

$$Z(k+1) = \frac{H}{k+1}\left[\sum_{l=0}^{k} X(l)Y(k-l) - bZ(k)\right]. \quad (20)$$

where the initial conditions are given by $X(0) = -10$, $Y(0) = 0$ and $Z(0) = 37$.

The difference equations presented in Eqs. (18)-(20) describe the Chen system, from a process of inverse differential transformation. It can be shown that [2] the solutions of each subdomain take $n+1$ terms for the power series like Eq. (11), i.e.,

$$x_i(t) = \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k X_i(k), \quad 0 \le t \le H_i, \quad (21)$$

$$y_i(t) = \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k Y_i(k), \quad 0 \le t \le H_i, \quad (22)$$

$$z_i(t) = \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k Z_i(k), \quad 0 \le t \le H_i, \quad (23)$$

where $k = 0, 1, 2, \ldots, n$ represents the number of terms of the power series, $i = 0, 1, 2, \ldots$ expresses the i-th subdomain and H_i is the subdomain interval.

4 Results and Discussions

RK4 has been widely and commonly used for simulating solutions for chaotic system. The Chen system can exhibit both chaotic and non-chaotic behaviour for distinct parameter values. In this paper, a comparison in the case of a fixed time step between the DTM and the RK4 for the solution of the chaotic Chen system are made. The accuracy of RK4 has to be determined first for the solution of (12)-(14) at different time steps before comparing with the DTM.

In this paper, the parameters $a = 35$ and $c = 28$ with $b = 12$ (nonchaotic) and $b = 3$ (chaotic) are fixed with the chosen initial point $x(0) = -10$, $y(0) = 0$ and $z(0) = 37$. The simulations done in this paper are for the time span $t \in [0,7]$ (see [2]). We decided to use 15 terms in the DTM series solutions based on our preliminary calculations.

Nonchaotic Solution

First, we considered the non-chaotic case where $a = 35$, $b = 12$ and $c = 28$. With the help of Maple 13, the approximate solutions are obtained from Eqs. (21)-(23) are as follows:

$$x(t) = -10 + 350t + 1575t^2 - \frac{186725}{3}t^3 + \frac{2008825}{4}t^4 + \frac{36773275}{12}t^5 - \frac{1805054335}{24}t^6 + \frac{13794025585}{72}t^7 + \frac{1307105651515}{192}t^8 - \frac{377410683279595}{5184}t^9 - \frac{2554510491404455}{10368}t^{10} + \frac{1240464104833247141}{114048}t^{11} - \frac{886742166509045146 99}{1368576}t^{12} - \frac{12640634830518546961543}{17791488}t^{13} + \frac{339219098335564141386331 3}{249080832}t^{14} \tag{30}$$

$$y(t) = 440t - 3760t^2 - \frac{14540}{3}t^3 + \frac{2819950}{3}t^4 - \frac{29486417}{3}t^5 - \frac{332044736}{9}t^6 + \frac{220205392501}{126}t^7 - \frac{857730678109}{72}t^8 - \frac{1299096318590405}{9072}t^9 + \frac{7993446094681189}{2520}t^{10} - \frac{3771497748478405397}{332640}t^{11} - \frac{851960628425335767 1}{25920}t^{12} + \frac{7374421910718730675523 27}{155675520}t^{13} - \frac{5599536595055866660531 9}{54486432}t^{14}, \tag{31}$$

$$z(t) = 37 - 444t + 464t^2 + \frac{186032}{3}t^3 - \frac{988996}{3}t^4 - \frac{40448516}{5}t^5 + \frac{809770482}{5}t^6 - \frac{2694568762}{5}t^7 - \frac{22830882548039}{1260}t^8 + \frac{977005325990431}{3780}t^9 - \frac{2338196538364391}{113400}t^{10} - \frac{2089996859048600023}{59400}t^{11} + \frac{5513541838281960395 69}{1663200}t^{12} + \frac{1096625843050020470601 91}{64864800}t^{13} - \frac{5503632119380119055647271 7}{908107200}t^{14}. \tag{32}$$

With the help of Maple 13, we have determined the solution points of (12)-(14) at different time steps by RK4. The solutions points obtained are presented in Table 1, Table 2 and Table 3. Besides, we have also determined the accuracy of RK4 first for the solution of (12)-(14) at different time steps before comparing with the DTM. Table 4 has presented the results of this analysis.

Table 1: The solutions points obtained by RK4 at $\Delta t = 0.01$ for $b = 12$.

t	x	y	z
1	-27.054660707952385	-25.829785688541233	37.195681864725281
2	-24.220203102041877	-15.960756261129872	38.727949536128537
3	-17.836012791172013	-5.014096669129172	35.030555096436205
4	-9.831210400585520	4.021272883035658	28.746019629584722
5	-1.930913832633712	10.866833989119137	22.890177162419099
6	5.185500567394349	16.603394521003989	19.140513211975744
7	11.616145579345145	22.122303983970380	18.221904191712490

Table 2: The solutions points obtained by RK4 at $\Delta t = 0.001$ for $b = 12$.

t	x	y	z
1	-27.047610997112465	-25.739174916436922	37.246933667368730
2	-24.094615765497160	-15.671791155115487	38.695923775424773
3	-17.530325686423151	-4.598760826330686	34.807847239447890
4	-9.383270776787116	4.451179921926637	28.396246353313699
5	-1.408563824663765	11.292701652193208	22.554860165609400
6	5.749554548317780	17.073991902763906	18.952855983564156
7	12.226017201470815	22.667499705337408	18.306715029396594

Table 3: The solutions points obtained by RK4 at $\Delta t = 0.0001$ for $b = 12$.

t	x	y	z
1	-27.047610303357526	-25.739166712852754	37.246938185625440
2	-24.094604480173528	-15.671765500479907	38.695920712709458
3	-17.530298572723827	-4.598724352586933	34.807827361975773
4	-9.383231385407043	4.451217410480568	28.396215623488707
5	-1.408518135446471	11.292738778094994	22.554831146374224
6	5.749603793217553	17.074033070884695	18.952840364482545
7	12.226070439195535	22.667547304917715	18.306723790831653

Table 4: A determination of accuracy of RK4 for $b = 12$.

t	$\Delta = \lvert RK4_{0.01} - RK_{0.001} \rvert$			$\Delta = \lvert RK4_{0.001} - RK_{0.0001} \rvert$		
	Δx	Δy	Δz	Δx	Δy	Δz
1	0.00705	0.09061	0.05125	6.938E-07	8.204E-06	4.518E-06
2	0.1256	0.289	0.03203	1.129E-05	2.565E-05	3.063E-06
3	0.3057	0.4153	0.2227	2.711E-05	3.647E-05	1.988E-05
4	0.4479	0.4299	0.3498	3.939E-05	3.749E-05	3.073E-05
5	0.5224	0.4259	0.3353	4.569E-05	3.713E-05	2.902E-05
6	0.5641	0.4706	0.1877	4.924E-05	4.117E-05	1.562E-05
7	0.6099	0.5452	0.08481	5.324E-05	4.760E-05	8.761E-06

Table 5: Differences between 15-term DTM and RK4 solutions for $b = 12$.

t	$\Delta = \lvert DTM_{0.05} - RK_{0.001} \rvert$			$\Delta = \lvert DTM_{0.01} - RK_{0.001} \rvert$		
	Δx	Δy	Δz	Δx	Δy	Δz
1	0.1236	1.057	0.5508	6.938E-07	8.204E-06	4.519E-06
2	0.6455	1.534	0.1605	1.129E-05	2.566E-05	3.063E-06
3	1.106	1.597	0.6777	2.712E-05	3.648E-05	1.988E-05
4	1.811	1.654	1.677	3.940E-05	3.749E-05	3.073E-05
5	1.706	1.487	0.9864	4.569E-05	3.713E-05	2.902E-05
6	1.762	1.476	0.5989	4.925E-05	4.117E-05	1.562E-05
7	1.752	1.472	0.04003	5.324E-05	4.760E-05	8.762E-06

Referring to Table 4, we could see that the maximum difference between the RK4 solutions on time steps $\Delta t = 0.001$ and $\Delta t = 0.0001$ is of the order of magnitude of 10^{-5}. This level of accuracy is matched by the 15-term DTM solutions on the smaller time step $\Delta t = 0.01$, as depicted in Table 5. Now realizing the potential of DTM method, a further step is done to demonstrate its accuracy at a smaller time step $\Delta t = 0.001$. We can see that the accuracy of the RK4 solutions is increasing by decreasing the time step, and this brought the DTM solutions and the RK4 solutions closer to each other up to a maximum difference of order $|10^{-8}|$ [2].

Chaotic Solution

Next, we considered the chaotic case where $a = 35$, $b = 3$ and $c = 28$. In this case the approximate solutions are obtained from Eqs. (21)-(23) are as follows:

$$\begin{aligned} x(t) = {} & -10 + 350t + 1575t^2 - \frac{128450}{3}t^3 + \frac{221725}{4}t^4 + \frac{11186560}{3}t^5 - \\ & \frac{754023445}{24}t^6 - \frac{4384159955}{18}t^7 + \frac{871980103135}{192}t^8 \\ & - \frac{24536076035}{648}t^9 - \frac{4580576044494961}{10368}t^{10} + \frac{4724308495578929}{1782}t^{11} + \\ & \frac{38904511962395575061}{1368576}t^{12} - \frac{136436333126299760 33}{31104}t^{13} - \\ & \frac{591081935565862940769 41}{249080832}t^{14}, \end{aligned} \tag{33}$$

$$\begin{aligned} y(t) = {} & 440t - 2095t^2 - \frac{109445}{3}t^3 + \frac{7057495}{12}t^4 - \frac{19884341}{12}t^5 - \frac{5769398299}{72}t^6 + \\ & \frac{400431583141}{504}t^7 + \frac{18272324444179}{4032}t^8 - \frac{654564294964703}{5184}t^9 + \\ & \frac{1578173135108086 9}{40320}t^{10} + \frac{14996049391664356 37}{120960}t^{11} - \\ & \frac{306785730292404817321}{2280960}t^{12} - \frac{33225373247543841425760 1}{622702080}t^{13} + \\ & \frac{167686111488364713419785541}{8717829120}t^{14}, \end{aligned} \tag{34}$$

$$z(t) = 37 - 111t - \frac{4067}{2}t^2 + \frac{362101}{6}t^3 + \frac{861097}{24}t^4 - \frac{327171397}{40}t^5 + \frac{4047783797}{80}t^6 + \frac{326595204709}{560}t^7 - \frac{64216557874249}{5760}t^8 + \frac{889264293069443}{120960}t^9 + \frac{4813574200599526613}{3628800}t^{10} - \frac{1878525121406975975 9}{1900800}t^{11} - \frac{4684354837870299463087}{53222400}t^{12} + \frac{36516369380802461676774 83}{2075673600}t^{13} - \frac{719695603546823833533360 7}{4151347200}t^{14}. \quad (35)$$

With the help of Maple 13, we have determined the solution points of (12)-(14) at different time steps by RK4 for chaotic Chen system. The solutions points obtained are presented at Table 6, Table 7 and Table 8. We expected the solutions are highly sensitive to time step since the system is chaotic. This is shown by the results presented in Table 9.

Table 6: The solutions points obtained by RK4 at $\Delta t = 0.01$ for $b = 3$.

t	x	y	z
1	-15.904279627723110	-13.162241166169599	32.088768779730344
2	16.163615171786290	6.128874885530317	40.622140400402537
3	-10.806055372478783	-8.570831723887125	30.730176224737862
4	7.376025873778008	13.998969329790345	36.104921021208311
5	0.100275676526658	2.188242498621259	20.390805267577554
6	-10.053647927618957	-11.334283841111115	25.252478725614482
7	15.739569434757931	16.623203918673357	26.963870881059763

Table 7: The solutions points obtained by RK4 at $\Delta t = 0.001$ for $b = 3$.

t	x	y	z
1	-15.904923862671938	-13.162222904435073	32.090047425873714
2	16.318177645974252	6.369459256345558	40.655950945086806
3	-10.705242534162903	-8.474052644673662	30.502572655311047
4	3.770851902703171	12.523956039625540	37.032553450438239
5	-1.751800326432415	2.618093565167990	25.330275929887352
6	-9.933292257336563	-10.295384284966910	23.242119839844348
7	-8.340718269094994	-8.336382626514890	25.087431073946013

Table 8: The solutions points obtained by RK4 at $\Delta t = 0.0001$ for $b = 3$.

t	x	y	z
1	-15.904923455907273	-13.162222060962183	32.090047601955776
2	16.318160368268374	6.369431543965352	40.6559479487732618
3	-10.705249232267281	-8.474056884447783	30.502592078905155
4	3.771509099784785	12.524292414387077	37.032301243234180
5	-1.751032483050321	2.618146534062463	25.328889429283835
6	-9.933262483349147	-10.296556942992606	23.240489345677946
7	-8.340787581856300	-8.342718784434861	25.075071051617045

Table 9: A determination of accuracy of RK4 for $b = 3$.

t	$\Delta = \lvert RK4_{0.01} - RK_{0.001}\rvert$			$\Delta = \lvert RK4_{0.001} - RK_{0.0001}\rvert$		
	Δx	Δy	Δz	Δx	Δy	Δz
1	0.0006442	1.826E-05	0.001279	4.068E-07	8.435E-07	1.761E-07
2	0.1546	0.2406	0.03381	1.728E-05	2.771E-05	2.996E-06
3	0.1008	0.09678	0.2276	6.698E-06	4.240E-06	1.942E-05
4	3.605	1.475	0.9276	0.0006572	0.0003364	0.0002522
5	1.852	0.4299	4.939	0.0007678	5.297E-05	0.001386
6	0.1204	1.039	2.01	2.977E-05	0.001173	0.00163
7	24.08	24.96	1.876	6.931E-05	0.006336	0.01236

Table 10: Differences between 15-term DTM and RK4 solutions for $b = 3$.

t	$\Delta = \lvert DTM_{0.01} - RK_{0.001}\rvert$			$\Delta = \lvert DTM_{0.001} - RK_{0.0001}\rvert$		
	Δx	Δy	Δz	Δx	Δy	Δz
1	4.068E-07	8.436E-07	1.761E-07	4.556E-11	9.305E-11	1.808E-11
2	1.728E-05	2.772E-05	2.997E-06	2.070E-09	3.314E-09	3.655E-10
3	6.699E-06	4.240E-06	1.943E-05	8.410E-10	5.601E-10	2.383E-09
4	0.0006573	0.0003364	0.0002522	8.048E-08	4.118E-08	3.091E-08
5	0.0007679	5.297E-05	0.001387	9.410E-08	6.420E-09	1.700E-07
6	2.978E-05	0.001173	0.001631	3.656E-09	1.441E-07	2.005E-07
7	6.932E-05	0.006337	0.01236	7.062E-09	7.779E-07	1.522E-06

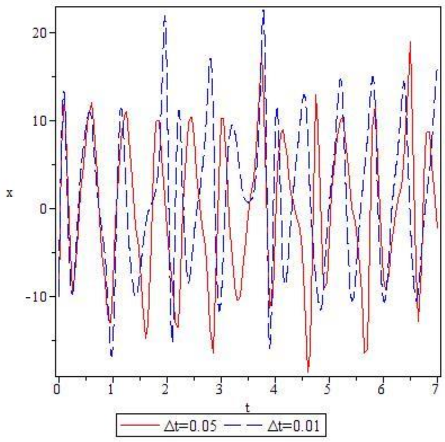

Figure 1: RK4 solutions on $\Delta t = 0.05$ vs $\Delta t = 0.01$ *t-x* plane.

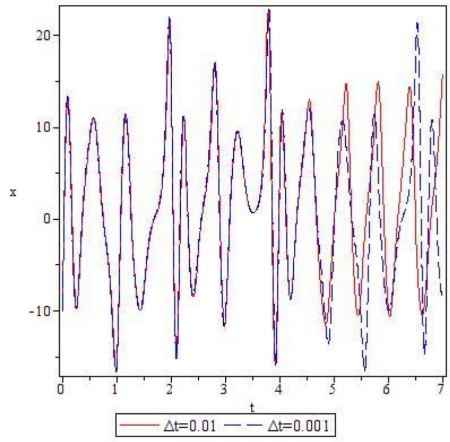

Figure 4: RK4 solutions on $\Delta t = 0.01$ vs $\Delta t = 0.001$ *t-x* plane.

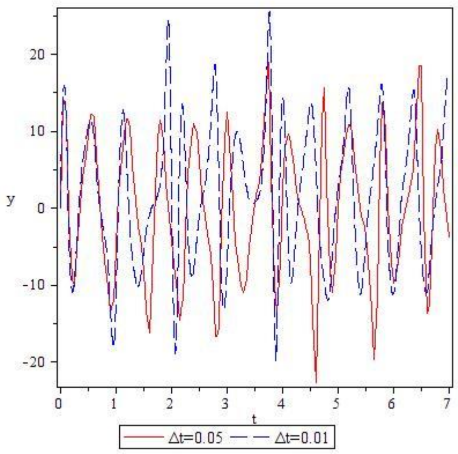

Figure 2: RK4 solutions on $\Delta t = 0.05$ vs $\Delta t = 0.01$ *t-y* plane

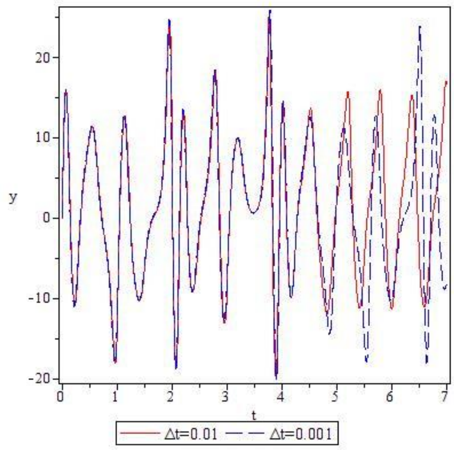

Figure 5: RK4 solutions on $\Delta t = 0.01$ vs $\Delta t = 0.001$ *t-y* plane.

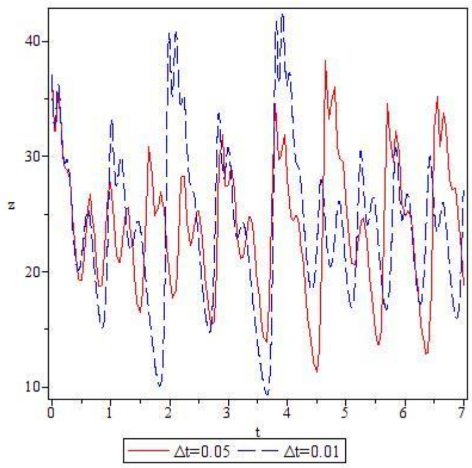

Figure 3: RK4 solutions on $\Delta t = 0.05$ vs $\Delta t = 0.01$ *t-z* plane.

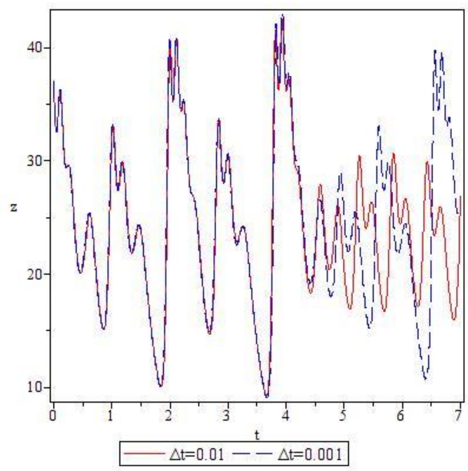

Figure 6: RK4 solutions on $\Delta t = 0.01$ vs $\Delta t = 0.001$ *t-z* plane.

RK4 solutions are presented separately because it is difficult to show clearly the RK4 solutions on various time steps on the same plot. From Figurse 1, 2 and 3, we can see that both the RK4 solution on $\Delta t = 0.05$ and $\Delta t = 0.01$ deviate when $t \geq 0.5$. While from Figures 4, 5 and 6, we can see that the solutions on $\Delta t = 0.01$ and $\Delta t = 0.001$ begin to deviate from each other when $t \geq 4$ (see also columns 2-4 of Table 8). The differences between the RK4 solutions on $\Delta t = 0.001$ and $\Delta t = 0.0001$ are given in the last three columns of Table 9. As expected, the solutions of the chaotic system become less accurate as time progresses. According to Table 9, we could see that the maximum difference between the RK4 solutions on time steps $\Delta t = 0.001$ and $\Delta t = 0.0001$ is of the order of magnitude of $|10^{-2}|$ which is much larger than in the nonchaotic case.

In the previous section, we have determined the 15-term DTM on time step $\Delta t = 0.001$ outperforms the RK4 on a much smaller step size $\Delta t = 0.0001$ for the nonchaotic case. The solutions of the chaotic system become less accurate as time progresses [2].

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References

[1] M.M. Al-Sawalha and M.S.M. Noorani, On solving the Lorenz system by differential transformation method, *Chin. Phys. Lett.*, **25** (2008), 1217 - 1219. http://dx.doi.org/10.1088/0256-307x/25/4/015

[2] M.M. Al-Sawalha and M.S.M. Noorani, A numeric-analytic method for approximating the Chen system, *Chaos, Solitons & Fractals,* **42** (2009), 1784-1791. http://dx.doi.org/10.1016/j.chaos.2009.03.096

[3] C.L. Chen and Y.C. Liu, Solution of two point boundary value problems using the differential transformation method, *J. Opt. Theory Appl.*, **99** (1998), 23-35.

[4] G. Chen and T. Ueta, Yet another chaotic attractor, *Int. J. Bifurcat. Chaos*, **9** (1999), no. 7, 1465-1466. http://dx.doi.org/10.1142/s0218127499001024

[5] W. Deng and C. Li, Synchronization of chaotic fractional Chen system, *J. Phys. Soc. Jap.*, **74** (2005), 1645-1648. http://dx.doi.org/10.1143/jpsj.74.1645

[6] S. Hussain and Z. Salleh, Continuous Galerkin Petrov time discretization scheme for the solutions of the Chen system, *J. Comput. Nonlinear Dynam.*, **10** (2015), no. 6, 061007. http://dx.doi.org/10.1115/1.4029714

[7] J. Lu, T. Zhou, G. Chen and S. Zhang, Local bifurcations of the Chen system, *Int. J. Bifurcat. Chaos*, **12** (2002), 2257-2270. http://dx.doi.org/10.1142/s0218127402005819

[8] S. Momani and M.A. Noor, Numerical comparison of methods for solving a special fourth-order boundary value problem, *Appl. Math. Comput.*, **191** (2007), 218-224. http://dx.doi.org/10.1016/j.amc.2007.02.081

[9] J.H. Park, Chaos synchronization of nonlinear Bloch equations, *Chaos, Solitons & Fractals*, **27** (2006), 357-361. http://dx.doi.org/10.1016/j.chaos.2005.03.047

[10] J.H. Park, Chaos synchronization between two different chaotic dynamical systems, *Chaos, Solitons & Fractals*, **27** (2006), 549-554. http://dx.doi.org/10.1016/j.chaos.2005.03.049

[11] T. Plienpanich, P. Niamsup and Y. Lenbury, Controllability and stability of the perturbed Chen chaotic dynamical system, *Appl. Math. Comput.*, **171** (2005), 927-947. http://dx.doi.org/10.1016/j.amc.2005.01.099

[12] T. Ueta and G. Chen, Bifurcation analysis of Chen's equation, *Int. J. Bifurcat. Chaos*, **10** (2000), 1917-1931. http://dx.doi.org/10.1142/s0218127400001183

[13] M.T. Yassen, Chaos control of Chen chaotic dynamical system, *Chaos, Solitons & Fractals*, **15** (2003), 271-283. http://dx.doi.org/10.1016/s0960-0779(01)00251-x

[14] L.T. Yu and C.K. Chen, Application of Taylor transformation to optimize rectangular fins with variable thermal parameters, *Appl. Math. Model*, **22** (1998), 11-21. http://dx.doi.org/10.1016/s0307-904x(98)00004-3

[15] J.K. Zhou, *Differential Transformation and its Application for Electrical Circuits,* Huarjung University Press, Wuhan, China, 1986 [in Chinese].

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